A REMARK ON α_G-INVARIANT

KEWEI ZHANG

Let X be a smooth n-dimensional Fano variety. Let $G \subset \mathrm{Aut}(X)$ be an arbitrary subgroup. Then the α_G-invariant of X is defined by

$$\alpha_G(X) = \sup \left\{ \lambda \in \mathbb{Q} \left| \begin{array}{l} \text{the log pair } \left(X, \dfrac{\lambda}{m}\mathcal{D}\right) \text{ is log canonical for every} \\ \text{G-invariant linear system } \mathcal{D} \subset |-mK_X| \text{ and } \forall m \geq 1 \end{array} \right. \right\}.$$

In this note, we prove the following

Proposition 0.1. *Suppose that $G = N \rtimes_\varphi \Gamma$, where $\varphi : \Gamma \to \mathrm{Aut}(N)$ is the group homomorphsim defining the semi-direct product between groups N and Γ. We assume that*

- *Γ is a finite group;*
- *for any N-invariant linear system $\mathcal{D} \subset |-mK_X|$, there always exists an N-invariant divisor $D \in \mathcal{D}$.*

Then we have

$$\alpha_G(X) = \sup \left\{ \lambda \in \mathbb{Q} \left| \begin{array}{l} \textit{the log pair } \left(X, \dfrac{\lambda}{m}D\right) \textit{ is log canonical for every} \\ G\textit{-invariant divisor } D \in |-mK_X| \textit{ and } \forall m \geq 1 \end{array} \right. \right\}.$$

Proof. Let $\mathcal{D} \subset |-mK_X|$ be any G-invariant linear system. So in particular, $\mathcal{D}$ is N-invariant. Then by assumption we can find an N-invariant divisor $D \in \mathcal{D}$.

Now for any $\gamma \in \Gamma$, we claim that $\gamma(D) \in \mathcal{D}$ is also N-invariant. Indeed, for any point $y \in \gamma(D)$, there exists $x \in D$ such that $y = \gamma(x)$. So for any $h \in N$, we have

$$h(y) = h(\gamma(x)) = \gamma\big(\varphi(\gamma^{-1})(h)(x)\big) \in \gamma(D).$$

Thus $\gamma(D)$ is N-invariant as well.

Now we put $\widetilde{D} := \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} \gamma(D)$. Then $\widetilde{D}$ is G-invariant and

$$\mathrm{lct}(X, \widetilde{D}) \leq \mathrm{lct}(X, \mathcal{D}).$$

So it is enough to look at G-invaraint divisors.

□

Example 0.2. *We take $G = \mathbb{C}^* \rtimes \mathbb{Z}_2$. Note that any $\mathbb{C}^*$-invariant linear system must have a one-dimensional sub-representation. So the assumption of the Proposition is satisfied.*