Analytic Thresholds, Canonical metrics, and Okounkov bodies

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Outline of my talk

- Motivation: canonical metrics
- Ding functional
- Analytic / Valuative thresholds
- Relation with Okounkov bodies

§ Motivation

- Kähler - Einstein problem

Let X be a cpt Kähler mfd of dim n.

Let ξ be a Käller class on X.

$$\boxed{\begin{array}{l}\text{Question: Is there } \omega^* \in \xi \text{ s.t.} \\ \qquad\qquad \mathrm{Ric}(\omega^*) = \lambda \omega^* \quad \text{for some } \lambda \in \mathbb{R}? \\ \qquad\qquad c_1(X) = \lambda \{\omega\}.\end{array}}$$

• The complex Monge – Ampere equation

(X, ω) be cpt Kähler, of dimension n.

$dd^c = \frac{\sqrt{-1}\,\partial\bar\partial}{2\pi}$.

Define $\mathcal{H}_\omega := \{ \varphi \in C^\infty(X, \mathbb{R}) \mid \omega_\varphi := \omega + dd^c\varphi > 0 \}$

Let dV be a smooth volume form on X.

The KE problem is related to the following equation:

$$\boxed{(\omega + dd^c\varphi)^n = e^{-\lambda\varphi}\, dV.} \qquad (*)_\lambda$$

- Known results for $(\omega + dd^c\varphi)^n = e^{-\lambda\varphi} dv$.

① $\lambda < 0$: Aubin, Yau, 70s.

② $\lambda = 0$: Yau, 70s (Calabi Conjecture)

③ $\lambda > 0$: There are obstructions!

Yau-Tian-Donaldson Conjecture: Solvability of $(*)_\lambda$ is related to certain "algebro-geometric stability" of the mfd (X, ω).

In what follows we assume $\lambda > 0$.

§ Ding functional

- Define for $\lambda > 0$

$$D^\lambda(\varphi) := -\frac{1}{\lambda}\log\int_X e^{-\lambda\varphi}\,dV - E(\varphi), \quad \varphi \in \mathcal{H}_\omega.$$

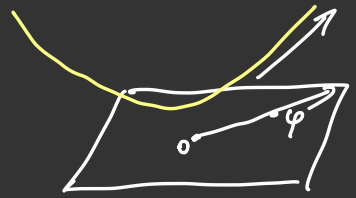
where $E(\varphi)$ is the Monge-Ampere energy (Aubin-Yau functional)

$$E(\varphi) = \frac{1}{(n+1)V}\int_X \varphi \sum_{i=0}^{n} \omega_\varphi^i \wedge \omega^{n-i}, \quad V = \int_X \omega^n.$$

★ If φ is a critical pt of D^λ, then

φ solves $\quad (\omega + dd^c\varphi)^n = e^{-\lambda\varphi}\,dV.$

• Properness



$\mathcal{H}_\omega$

$\sup_X \varphi = 0.$ $\quad -E(\varphi) = \underline{d_1}(\varphi, 0).$

Darvas distance

We say D^λ is proper/coercive if $\exists\, \varepsilon > 0,\ C > 0$ s.t.

$$D^\lambda(\varphi) \geq \varepsilon\left(\sup \varphi - E(\varphi)\right) - C \quad \text{for } \forall \varphi \in \mathcal{H}_\omega.$$

▲ <u>Thm</u> (BEGZ variational principle + Szekelyhidi–Tosatti regularity)

If D^λ is proper, then $(*)_\lambda$ admits a C^∞ solution

§ Analytic Thresholds

Moser-Trudinger type inequality.

- Tian's α-invariant

$$\alpha(\{\omega\}) = \sup\left\{ \lambda > 0 \;\middle|\; \sup_{\varphi \in \mathcal{H}_\omega} \int_X e^{-\lambda(\varphi - \sup\varphi)}\, dV < +\infty \right\}.$$

It only depends on the Kähler class $\{\omega\}$, and $\underline{\alpha(\{\omega\}) > 0}$.

- Tian's criterion:

$$\boxed{D^\lambda \text{ is proper for } \lambda < \frac{n+1}{n}\alpha(\{\omega\}).}$$

$\Longrightarrow$ For $\lambda < \frac{n+1}{n}\alpha$, one can solve $(*)_\lambda$.

- The (analytic) δ-invariant (Z, 2020)

$$\delta^A(\{\omega\}) = \sup\left\{\lambda>0 \;\middle|\; \sup_{\varphi\in\mathcal{H}_\omega} \int_X e^{-\lambda(\varphi - E(\varphi))}\, dV < +\infty\right\}$$

$\delta^A = 1 \Rightarrow X$ is K-semistable, $D^{\lambda=1} \geq -C$.

- $\boxed{\text{Prop (Tian-Zhu, Phong-Song-Sturm-Weinkove, BBEGZ) } D^\lambda \text{ is proper iff } \delta^A > \lambda}$

- Fact: $\boxed{\delta^A \geq \frac{n+1}{n}\alpha.}$ ← The proof is relatively easy, by using Jensen's inequality and Calabi-Yau theorem.

This explains why Tian's criterion holds.

$X = \mathbb{CP}^n$.
$\xi = c_1(-K_X)$.
Then $\alpha = \frac{1}{n+1}$, $\delta = \delta^A = 1$.
$\frac{1}{n+1}$ is optimal.

- Question (Z, 2020): $\boxed{\text{Do we have } \alpha \geq \frac{1}{n+1}\delta^A \text{ ?}}$

This is known to hold if $\omega \in c_1(L)$, L ample. (K. Fujita)

§ Valuative Thresholds.

- Let $Y \xrightarrow{\pi} X$ be a proper bimeromorphic morphism and $E \subseteq Y$ be a prime divisor (reduced, irreducible of codim 1).

  Such E is called a divisor over X.

  E induces a valuation ord_E on $K(X)$:

  For a meromorphic function f on X, can measure the order of zero/pole of f along E.

- There are several "functionals" associated to $E \subseteq Y \xrightarrow{\pi} X$

  Let ξ be a Kähler class on X.

Log discrepancy: $A_X(E) := 1 + \mathrm{ord}_E(K_Y - \pi^* K_X)$

pseudoeffective threshold: $T_\xi(E) = \sup\{ x > 0 \mid \pi^*\xi - xE \text{ big} \}$.

expected Lelong number: $S_\xi(E) = \frac{1}{\mathrm{Vol}(\xi)} \int_0^{T_\xi(E)} \mathrm{Vol}(\pi^*\xi - xE)\, dx$

- Rmk These notions are firsted defined by algebraic geometers for projective mfds, but they make sense for Kähler mfds as well.

- The valuative formulation of α-invariant

Prop: Let ξ be a Kähler class on X, then

$$\underbrace{\alpha(\xi)}_{\sup\{\lambda>0:\int e^{-\lambda\varphi}<+\infty\}=\inf_{E/X}\frac{A_X(E)}{\nu(\varphi,E)}} = \inf_{E/X} \frac{A_X(E)}{\tau_\xi(E)}$$

BFJ + GZ.

Rmk: When $\xi = c_1(L)$ for L ample, this was due to Demailly. The general case follows easily if one uses the valuative criterion of Boucksom-Favre-Jonsson.

- The valuative δ- invariant:

$$\boxed{\delta(\xi) := \inf_{E/X} \frac{A_X(E)}{S_\xi(E)}}$$

$|mL|$

$\delta_m \xrightarrow{m\to\infty} \delta$

- This invariant was first introduced by Fujita-Odaka in the Fano case and later further polished by Blum-Jonsson in the projective case.

- Conjecture (Z, 2020): One has $\delta^A(\xi) = \delta(\xi)$ for $\forall$ Kähler class.

BBJ.[18] CRZ.[18]

If X is Fano, $\xi = c_1(-K_X)$. And $\delta(-K_X) \le 1$.

Then $\delta(-K_X) = \delta^A(-K_X) =$ greatest Ricci lower bound

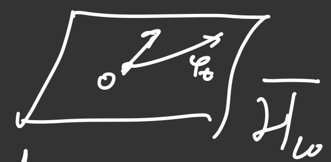
- Known results $\quad \mathcal{H}_\omega \xleftarrow{\text{Tian}} \mathcal{B}_m \quad m \to \infty.$

$$\mathcal{B}_m \downarrow \quad \delta_m = \delta_m^A \quad m \to \infty$$

① (Z, 2021) $\boxed{\delta^A(\xi) = \delta(\xi)}$ holds if $\xi = c_1(L)$

The proof relies on quatization methods going back to Tian.

② (Darvas-Z, 2022)



For general Kähler class ξ, one has

$$\boxed{\delta(\xi) = \sup\left\{ \lambda > 0 \;\middle|\; \lim_{t\to\infty} \frac{D^\lambda(\varphi_t)}{t} \geq 0 \text{ for } \forall \text{ geodesic ray } \varphi_t \right\}}$$

This implies : $\boxed{\delta^A(\xi) \leq \delta(\xi)}$

$\Downarrow$

If X is Fano and $\delta(-K_X) > 1$ then X admits KE.

The proof relies on pluripotential theory

• Consequence : We have an affirmative answer to the question :

$$\boxed{\alpha(\xi) \geq \frac{1}{n+1} \delta^{A}(\xi)}$$

$\delta^A \leq \delta$

Proof : It suffices to show

$$\alpha(\xi) \geq \frac{1}{n+1} \delta(\xi)$$

$\inf_E \frac{A}{T} \overset{?}{\geq} \frac{1}{n+1} \inf_E \frac{A}{S}$

It is enough to argue that

$$S_\xi(E) \geq \frac{1}{n+1} T_\xi(E).$$

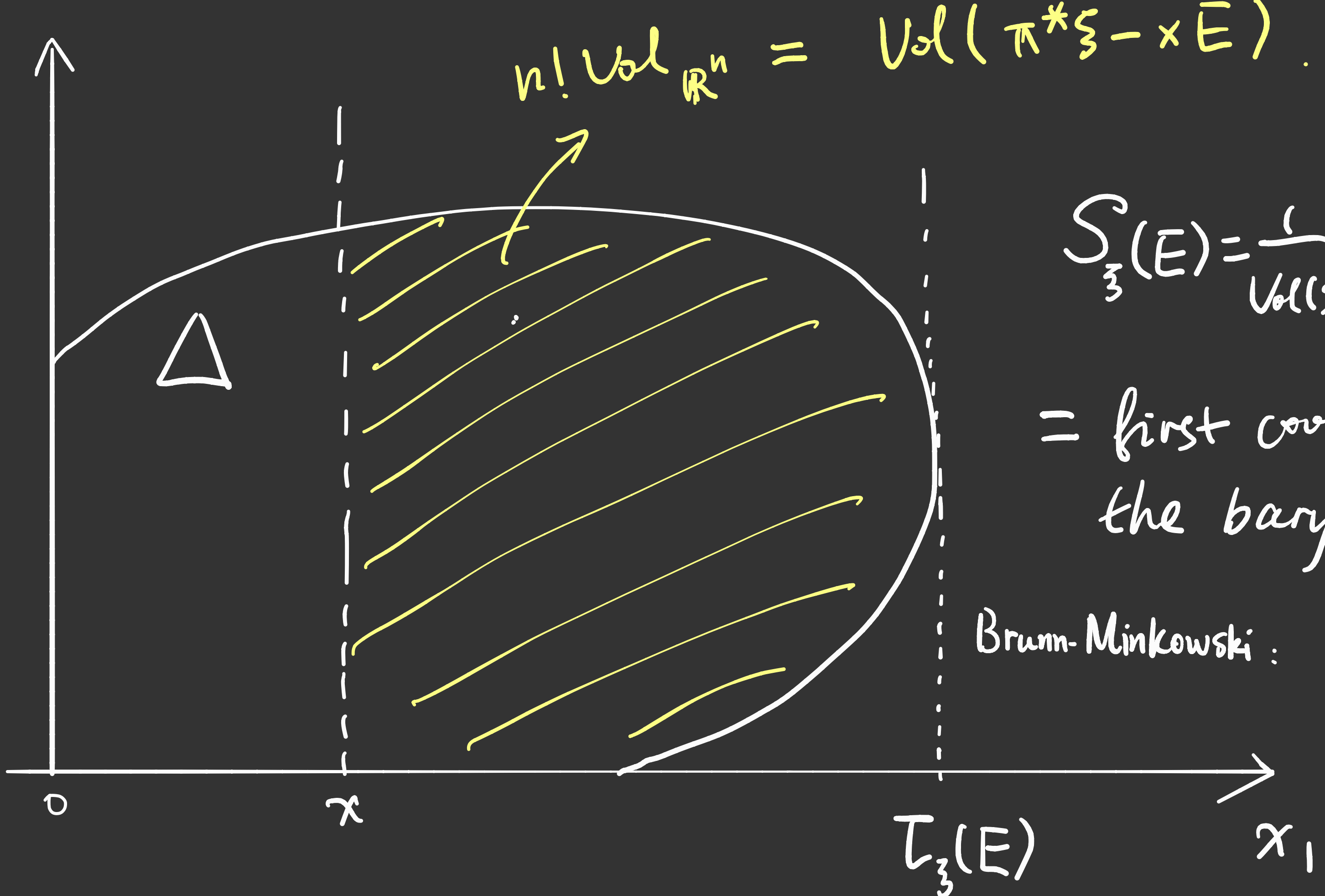
When $\xi = c_1(L)$, this was observed by K. Fujita.

- One can compare S and T using Okounkov bodies!

- In our recent work [Darvas-Reboulet-Witt Nyström-Xia-Z] we established a theory of transcendental Okounkov bodies, which associates a convex body $\Delta \subseteq \mathbb{R}^n$ to a big class ξ s.t.

$$\mathrm{Vol}(\xi) = n! \ \mathrm{Vol}_{\mathbb{R}^n}(\Delta)$$

- Given E over X, can construct Okounkov body Δ in such a way:



$$S_\xi(E) = \frac{1}{\mathrm{Vol}(\xi)}\int_0^{T_\xi(E)} \mathrm{Vol}(\pi^*\xi - xE)\,dx$$

= first coordinate of the barycenter of Δ

Brunn-Minkowski: $S_\xi(E) \geq \frac{1}{n+1} T_\xi(E)$.

Thanks for your attention!