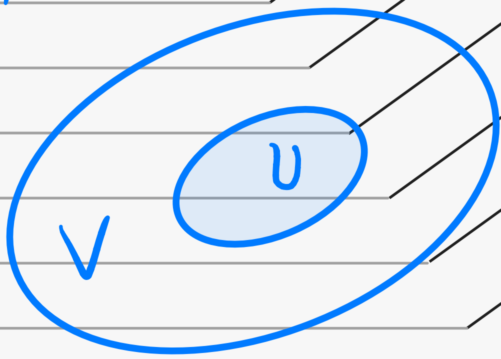
复几何课程

Lecture 1. Holomorphic functions of several complex variables.

$\bar{\partial} f = 0$

$f(z_1, \cdots, z_n) = \sum c_\alpha z^\alpha$



Outline (2 hours)

1. ∂ & $\bar\partial$
2. Cauchy integral formula < dim 1 / higher dim
3. Holomorphic functions and their properties
   - Power series expansion
   - Weierstrass's convergence thm
   - Cauchy's ineq.
   - Max principle
   - Identity theorem
   - Liouville's theorem (exam) — $\forall$ bounded $f \in \mathcal{O}(\mathbb{C}^n)$ must be const.
4. Hartogs Theorem (Prove by solving $\bar\partial$-equ)

- 1. ∂ & $\bar\partial$

Let $\Omega \subseteq \mathbb{C}^n$ be a domain (open & connected subset in $\mathbb{C}^n$)

Let $(z^1, \cdots, z^n)$ be the standard complex coordinates of $\mathbb{C}^n$.

Put $z^i = x^i + \sqrt{-1}\, y^i$.

▲ Define ∂ & $\bar\partial$ as follows.

$$\begin{cases} \dfrac{\partial}{\partial z^i} := \dfrac{1}{2}\left(\dfrac{\partial}{\partial x^i} - \sqrt{-1}\dfrac{\partial}{\partial y^i}\right) \\[2ex] \dfrac{\partial}{\partial \bar z^i} := \dfrac{1}{2}\left(\dfrac{\partial}{\partial x^i} + \sqrt{-1}\dfrac{\partial}{\partial y^i}\right). \end{cases} \quad \& \quad \begin{cases} dz^i = dx^i + \sqrt{-1}\, dy^i \\ d\bar z^i = dx^i - \sqrt{-1}\, dy^i \end{cases} \quad \text{for } \forall\ 1 \le i \le n.$$

Then
$$\begin{cases} \partial := \sum_{i=1}^{n} \dfrac{\partial}{\partial z^i} \otimes dz^i \\[2ex] \bar\partial := \sum_{i=1}^{n} \dfrac{\partial}{\partial \bar z^i} \otimes d\bar z^i \end{cases}$$

More concretely, for $\forall\, f \in C^1(\Omega, \mathbb{C})$, we have

$$\begin{cases} \partial f = \sum_i \frac{\partial f}{\partial z^i} dz^i = \frac{1}{2}\sum_i \left(\frac{\partial f}{\partial x^i} dx^i + \frac{\partial f}{\partial y^i} dy^i + \sqrt{-1}\frac{\partial f}{\partial x^i} dy^i - \sqrt{-1}\frac{\partial f}{\partial y^i} dx^i \right). \\ \bar\partial f = \sum_i \frac{\partial f}{\partial \bar z^i} d\bar z^i = \frac{1}{2}\sum_i \left(\frac{\partial f}{\partial x^i} dx^i + \frac{\partial f}{\partial y^i} dy^i + \sqrt{-1}\frac{\partial f}{\partial y^i} dx^i - \sqrt{-1}\frac{\partial f}{\partial x^i} dy^i \right) \end{cases}$$

▲ Observe that

$$\partial f + \bar\partial f = df = \sum_i \left(\frac{\partial f}{\partial x^i} dx^i + \frac{\partial f}{\partial y^i} dy^i \right).$$

Namely, $\boxed{d = \partial + \bar\partial}$

☆ ▲ We call $f \in C^1(\Omega, \mathbb{C}^n)$ holomorphic if f satisfies $\bar\partial f = 0$, ie if we write $f = u + \sqrt{-1}v$ then

$$\sum_i \left(\frac{\partial u}{\partial x^i} dx^i + \sqrt{-1}\frac{\partial v}{\partial x^i} dx^i + \frac{\partial u}{\partial y^i} dy^i + \sqrt{-1}\frac{\partial u}{\partial y^i} dy^i + \sqrt{-1}\frac{\partial u}{\partial y^i} dx^i - \frac{\partial v}{\partial y^i} dx^i - \sqrt{-1}\frac{\partial u}{\partial x^i} dy^i + \frac{\partial v}{\partial x^i} dy^i \right) = 0$$

$$\Leftrightarrow \begin{cases} \frac{\partial u}{\partial x^i} = \frac{\partial v}{\partial y^i} \\ \frac{\partial u}{\partial y^i} = -\frac{\partial v}{\partial x^i} \end{cases} \quad \text{for } \forall\ 1 \le i \le n.$$

This is called <u>Cauchy-Riemann equation</u>.

In particular, if f is holomorphic, then f is holomorphic in each complex variable (as a one-variable holomorphic/analytic function).

• 2 Cauchy integral formula.

- When $n=1$, assume that Ω is a bounded open set in $\mathbb{C}$ s.t. $\partial\Omega$ consists of finitely many C^1 Jordan curves. Then for $\forall\ u \in C^1(\bar{\Omega})$ we have

$$u(z_0) = \frac{1}{2\pi i}\left(\int_{\partial\Omega} \frac{u(z)}{z-z_0}\,dz + \iint_{\Omega} \frac{\partial u/\partial \bar{z}}{z-z_0}\, dz \wedge d\bar{z}\right) \text{ for } \forall\ z_0 \in \Omega.$$

pf: By definition

$$\iint_{\Omega} \frac{\frac{\partial u}{\partial \bar{z}} dz\wedge d\bar{z}}{(z-z_0)} = \lim_{\varepsilon\to 0} \iint_{\Omega\setminus B_\varepsilon(z_0)} \frac{\frac{\partial u}{\partial \bar{z}} dz\wedge d\bar{z}}{(z-z_0)}$$

$$= \lim_{\varepsilon\to 0} \iint_{\Omega\setminus B_\varepsilon(z_0)} \frac{-\bar{\partial}(u\,dz)}{(z-z_0)} = \lim_{\varepsilon\to 0} \iint_{\Omega\setminus B_\varepsilon(z_0)} \frac{-d(u\,dz)}{(z-z_0)}$$

$$= \lim_{\varepsilon\to 0}\left(\int_{\partial B_\varepsilon(z_0)} \frac{u\,dz}{z-z_0} - \int_{\partial\Omega} \frac{u\,dz}{z-z_0}\right)$$

$$= \lim_{\varepsilon\to 0} \int_{\partial B_\varepsilon(z_0)} \frac{u(z_0)\,dz}{z-z_0} - \int_{\partial\Omega} \frac{u\,dz}{z-z_0}$$

$$= 2\pi\sqrt{-1}\,u(z_0) - \int_{\partial\Omega} \frac{u\,dz}{z-z_0}$$

This completes the proof. $\square$

▲ The above formula has several consequences.

$n=1$

① When f is holomorphic ($n=1$) one has
$$f(z_0)=\frac{1}{2\pi i}\int_{\partial\Omega}\frac{f(z)}{z-z_0}\,dz$$
, which is the classical Cauchy integral formula.

② When $f\in C_0^1(\Omega)$, then
$$f(z_0)=\frac{1}{2\pi i}\iint_{\Omega}\frac{\partial f/\partial\bar z}{z-z_0}\,dz\wedge d\bar z$$

poly disk

$n\geq 1$ — ③ For general $n\geq 1$, if f is holomorphic on $\{(z_1,\dots,z_n)\in\mathbb{C}^n \mid |z_i|<r_i,\ i=1,\dots,n\}=:\mathbb{D}_r$

then
$$f(\xi)=\left(\frac{1}{2\pi i}\right)^n\int_{|z^1|=r_1}\cdots\int_{|z^n|=r_n}\frac{f(z^1,\dots,z^n)}{(z^1-\xi^1)\cdots(z^n-\xi^n)}\,dz^1\cdots dz^n$$

$r=(r_1,\dots,r_n)\in(\mathbb{R}_{>0})^n$

for $\forall\ \xi=(\xi^1,\dots,\xi^n)\in\mathbb{D}_r$

This is the Cauchy integral formula for holomorphic functions of several cplx varb.

• Def: We put $\mathcal{O}(\Omega):=\{$ holomorphic function on $\Omega\}$.

• <u>Thm</u>. The following are equivalent

① $f \in \mathcal{O}(\Omega)$

② f satisfies the Cauchy integral formula for $\forall$ polydisk $\mathbb{D}_r \subseteq \Omega$.

③ For $\forall\, z_0 \in \Omega$, $\exists$ polydisk $\mathbb{D}_r$ around z_0 s.t.

$$f(z) = \sum_{\nu \in \mathbb{N}^n} a_\nu (z - z_0)^\nu$$ namely f has a power series expansion

$$\hookrightarrow a_\nu = \frac{1}{\nu_1! \cdots \nu_n!} \frac{\partial^{|\nu|} f}{\partial z_1^{\nu_1} \cdots \partial z_n^{\nu_n}}(z_0) =: D^\nu f(z_0).$$

• <u>Rmk</u>. In ③, a_ν is given by

$$a_\nu = \left(\frac{1}{2\pi i}\right)^n \int_{|z^1 - z_0^1| = r_1} \cdots \int_{|z^n - z_0^n| = r_n} \frac{f(z^1, \cdots, z^n)}{(z^1 - z_0^1)^{\nu_1 + 1} \cdots (z^n - z_0^n)^{\nu_n + 1}} \, dz^1 \cdots dz^n$$

which is indpd of the choice of polydisk as long as $|r| \ll 1$.

This follows from the taylor expansion

$$\frac{1}{(\zeta - z)} = \frac{1}{\zeta - z_0 - (z - z_0)} = \frac{1}{(\zeta - z_0)} \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \frac{1}{(\zeta - z_0)} \sum_\nu \left(\frac{z - z_0}{\zeta - z_0}\right)^\nu$$

• <u>Cor</u>. If $\begin{cases} f(z) = \sum a_\nu z^\nu \\ g(z) = \sum b_\nu z^\nu \end{cases}$ two convergent power series around $0 \in \mathbb{C}^n$

& if $f = g$ over some small nbhd $V \subseteq \mathbb{C}^n$ containing 0

then $a_\nu = b_\nu$.

(Since a_ν & b_ν are determined by the infinitesimal information of f & g around 0)

• **Thm** Weierstrass's Convergence theorem.

Let $\{f_k\} \subseteq \mathcal{O}(\Omega)$ be a sequence of hol. functions on Ω that converges uniformly to a function f. Then $f \in \mathcal{O}(\Omega)$.

pf: $f = \lim f_k = \lim_k \int\!\!\int \frac{f_k}{(\zeta - z)} d\zeta = \int \cdots \int \frac{f}{(\zeta - z)} d\zeta =$ has power series expansion. $\square$

• **Thm**. For $f_1, f_2 \in \mathcal{O}(\Omega)$, assume that for some $U \subseteq \Omega$

$f_1|_U = f_2|_U$. then $f_1 = f_2$.

pf: Put $N := \{ z_0 \in \Omega$ s.t. $D^\nu f_1(z_0) = D^\nu f_2(z_0)$ for $\forall \nu \in \mathbb{N}^n \}$.

Then N is clearly closed. & $U \subseteq N$.

N is also open as $D^\nu f_1(z_0) = D^\nu f_2(z_0)$ implies that $f_1 = f_2$ around z_0.

Thus we must have $N = \Omega$. $\square$

• **Thm** (Max principle). If $f \in \mathcal{O}(\Omega)$ & $\exists z_0 \in \Omega$ s.t.

$|f|$ is locally maximized at z_0, then f is constant.

pf 1. Consider complex lines through z_0 & use max principle of 1-variable hol. fnct.

pf 2. Using mean value formula of f & using the fact that $|f| = \text{Const} \Rightarrow f = \text{Const}$.

$\partial f = 0 \Rightarrow df = 0$

exam

• **Thm** (Hartogs thm) Assume that $n \geq 2$.

Let Ω be a domain & $K \subseteq \Omega$ a compact subset s.t. $\Omega \setminus K$ connected.
Then $\forall f \in \mathcal{O}(\Omega \setminus K)$ can be extended to a function $\hat{f} \in \mathcal{O}(\Omega)$
s.t. $f = \hat{f}$ on $\Omega \setminus K$.

▲ This is obviously not true when $n = 1$.

pf: Choose a cut-off function $\varphi \in C_0^\infty(\Omega)$ s.t.
$\varphi \equiv 1$ on a nbhd of K. Consider
$f_0 := (1-\varphi) f$. Then $f_0 \in C_0^\infty(\mathbb{C}^n)$
Put $\alpha := \bar{\partial} f_0 = -f \bar{\partial}\varphi$, which is a C_0^∞-$(0,1)$ form.
Obviously, $\bar{\partial}\alpha = 0$. If write $\alpha = \sum_{i=1}^n \alpha_i \, d\bar{z}_i$
then $\frac{\partial \alpha_i}{\partial \bar{z}_j} = \frac{\partial \alpha_j}{\partial \bar{z}_i}$ for $\forall i, j$.

Put
$$u(z) = \frac{1}{2\pi i} \iint_\Omega \frac{\alpha_1(\tau, z_2, \cdots, z_n)}{(\tau - z_1)} d\tau \wedge d\bar{\tau} \quad \text{for } \forall z \in \Omega.$$
$$= \frac{1}{2\pi i} \iint_\Omega \frac{\alpha_1(\tau + z_1, z_2, \cdots, z_n)}{\tau} d\tau \wedge d\bar{\tau}$$
So in particular $u \in C_0^\infty$ & we have
$$\frac{\partial u(z)}{\partial \bar{z}_k} = \frac{1}{2\pi i} \iint_\Omega \frac{\partial \alpha_k(\tau, z_2, \cdots, z_n)/\partial \bar{\tau}}{(\tau - z_1)} d\tau \wedge d\bar{\tau} = \alpha_k(z) \text{ by Cauchy integral formula.}$$

Thus u solves $\bar{\partial} u = \alpha$. $\Rightarrow \bar{\partial}(f_0 - u) = 0$

$\Rightarrow \hat{f} := f_0 - u \in \mathcal{O}(\Omega)$

Notice that $\begin{cases} u = 0 \\ f_0 = f \end{cases}$ on an open subset of $\Omega \setminus K$, thus $\hat{f} = f$ on an open subset of $\Omega \setminus K$ & hence $\hat{f} = f$ on $\Omega \setminus K$ (as $\Omega \setminus K$ is connected)

This completes the proof. $\square$.

- We end this lecture by giving the definition of meromorphic functions.

  f is called a meromorphic function on Ω if $\exists$ open cover $\Omega = \bigcup U_i$ and $f_i, g_i \in \mathcal{O}(U_i)$ s.t. $f = \frac{f_i}{g_i}$ on U_i