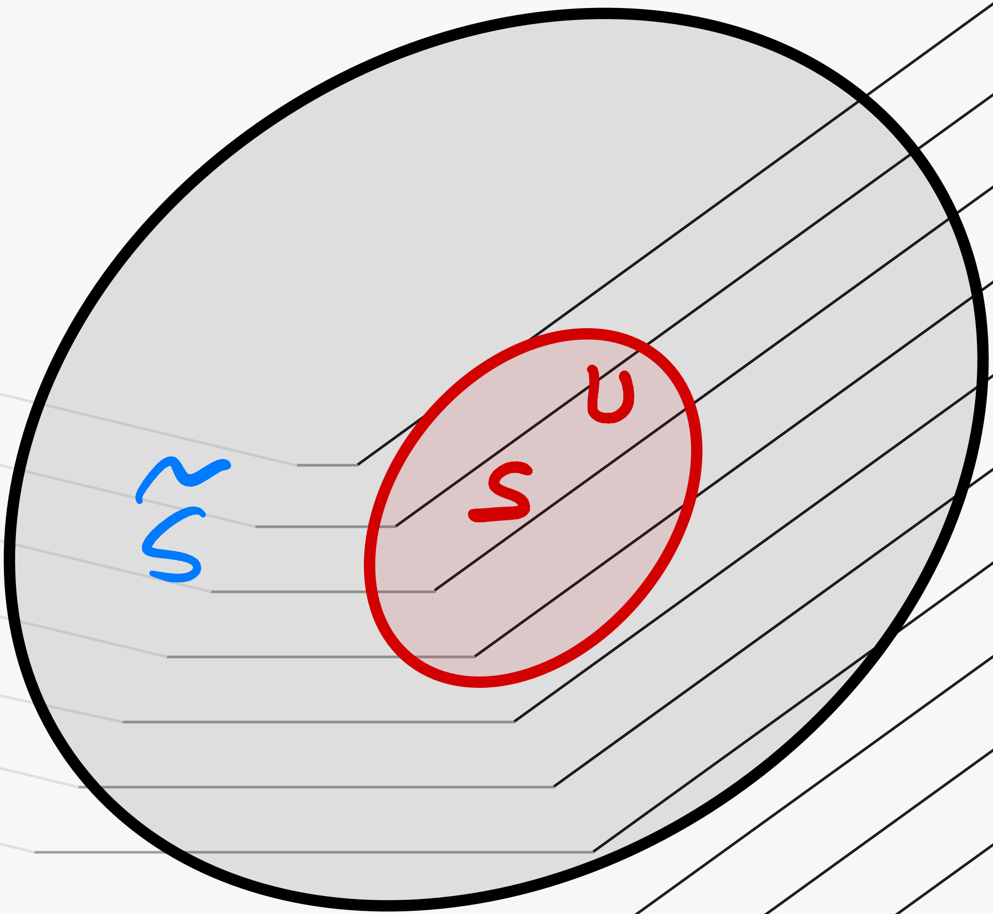
Lecture 3

Sheaf cohomology.



$$s = \tilde{s}\big|_U$$

Outline ① exact sequence & examples.

② resolution of sheaves

③ cohomology & examples.

- Recall Given $\varphi: \mathcal{F} \to \mathcal{G}$

$$\begin{cases} ① \ \varphi \text{ is inj.} \\ ② \ \varphi_x \text{ is inj. for } \forall x \in X \\ ③ \ \varphi_U : \mathcal{F}(U) \to \mathcal{G}(U) \text{ inj. for } \forall U. \end{cases}$$

$$\begin{cases} ① \ \varphi \text{ is surj.} \\ ② \ \varphi_x \text{ is surj. for } \forall x \in X. \\ ③ \ \forall \tau \in \mathcal{G}(U) \ \exists \text{ open cover } U = \cup U_i \ \& \ s_i \in \mathcal{F}(U_i) \text{ s.t. } \tau|_{U_i} = \varphi_{U_i}(s_i). \end{cases}$$

- We call $0 \to \mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{C} \to 0$ exact

if α inj, β surj. & $\ker\beta = \operatorname{Im}\alpha$.

i.e. $0 \to \mathcal{A}_x \xrightarrow{\alpha_x} \mathcal{B}_x \xrightarrow{\beta_x} \mathcal{C}_x \to 0$ is an exact sequence.

- e.g. ① let $X := \mathbb{C}$. $\mathcal{A} := \mathbb{Z}$, $\mathcal{B} := \mathcal{O}_X$ & $\mathcal{C} := \mathcal{O}_X^*$.

$\mathcal{A}$: constant sheaf. $\mathcal{B}$:= sheaf of holomorphic functions

$\mathcal{C}$:= sheaf of holomorphic functions that is no-where vanishing

then we have $0 \to \mathbb{Z} \xrightarrow{i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0$.

Verify that this is exact.

② If $\mathcal{A}$ is a subsheaf of $\mathcal{B}$, then

$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{A}/\mathcal{B} \to 0$ is exact.

③ e.g. $X := \mathbb{C}$, $\mathcal{B} := \mathcal{O}_X$. $\mathcal{A} := \mathcal{I}_0$ ideal sheaf of 0.

then $\left(\mathcal{B}/\mathcal{A}\right)_x = \begin{cases} \mathbb{C}, & x=0 \\ 0, & x \neq 0. \end{cases}$ this is a skyscraper sheaf.

★ ④ $X = \mathbb{C}^2$, $Y := \{z_1 = 0\}$ let $\mathcal{I}_Y :=$ sheaf of holo. functions vanish along Y.

then $0 \to \mathcal{I}_Y \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}_Y \to 0$.

Observe that $\mathcal{O}_X/\mathcal{I}_Y \cong \mathcal{O}_Y$.

This exact sequence gives information for X & Y.

- More generally, we say
$$\cdots \to \mathcal{F}^0 \xrightarrow{d_0} \mathcal{F}^1 \xrightarrow{d_1} \mathcal{F}^2 \xrightarrow{d_2} \cdots$$
is a complex of sheaves if $d_{i+1} \circ d_i = 0$.

We say $0 \to \mathcal{F} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \cdots$ is a resolution of $\mathcal{F}$ if this complex is exact.

- e.g. let M be a Riem mfd.

$\Omega^i :=$ sheaf of C^∞ i-forms on M.

then $\Omega^0 =$ sheaf of C^∞ functions on M.

Then $0 \to \mathbb{Z} \xrightarrow{i} \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \cdots$ is a resolution of the constant sheaf $\mathbb{Z}$.

& $0 \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \cdots$ is a complex.

it is not exact at Ω^0.

exactness at Ω^i for $i \geq 1$ follows from Poincaré Lemma

- e.g. M C^∞ mfld. For $\forall$ U open let

$S_p(U) := \{ C^\infty$ chains : linear combinations of C^∞ maps $f : \Delta^p \to U \}$
C^∞ in a nbhd of Δ^p.

Note that $\{S_p(U)\}_U$ doesn't give rise to sheaves as there is no naturation map $S_p(U) \to S_p(V)$ whenever $V \subseteq U$. But there is indeed an inclusion $S_p(V) \hookrightarrow S_p(U)$ so it induces a restriction

$$\mathrm{Hom}(S_p(U), \mathbb{R}) \longrightarrow \mathrm{Hom}(S_p(V), \mathbb{R}).$$

Let $S^p(U) := \mathrm{Hom}(S_p(U), \mathbb{R})$.

Then $\{S^p(U)\}_U$ actually gives rise to a sheaf on M.

The usual coboundary map $\delta : S^p \to S^{p+1}$ gives a resolution of the constant sheaf $\mathbb{R}$.

$$0 \to \mathbb{R} \xrightarrow{i} S^0 \xrightarrow{\delta} S^1 \to S^2 \to \cdots$$

for each pt $x \in X$ associate it to a constant value.

- Now, what is cohomology?

Given a short exact sequence

$$0 \to \mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{C} \to 0$$

then it is straightforward to show that the sequence
e.x.

$$0 \to \mathcal{A}(X) \to \mathcal{B}(X) \to \mathcal{C}(X) \to 0$$

is exact at $\mathcal{A}(X)$ & $\mathcal{B}(X)$ but not at $\mathcal{C}(X)$.

The cohomology group measures the inexactness!

- Another example.

① Given a surjection between two groups

$$\varphi : G \to H.$$

For another group K, it is easy to see that φ induces an injection $\varphi^*: \mathrm{Hom}(H,K) \to \mathrm{Hom}(G,K)$.

② Now given an injection $\varphi: G \hookrightarrow H$

it is not necessarily true that

$\varphi^*: \mathrm{Hom}(H,K) \to \mathrm{Hom}(G,K)$ is surjection.

So in general there are obstructions for a morphism $G \to K$ to extend to a morphism $H \to K$.

③ So in general, whether something can be extended or whether something is the restriction of something from a bigger space are both questions about "obstructions".

These are often related to cohomology theory!

- Axioms of sheaf cohomology.

Let X be a paracompact Hausdorff space X. (so X has partition of unity)

↳ $\forall$ open cover admits a locally finite refinement.

Then for $\forall$ sheaf $\mathcal{F}$ of abelian groups one can associate a sequence of groups $H^q(X, \mathcal{F})$ for $q \geq 0$ s.t.

① $H^0(X, \mathcal{F}) = \mathcal{F}(X)$

② If $\mathcal{F}$ soft, then $H^q(X, \mathcal{F}) = 0$ for all $q > 0$

③ For $\forall$ sheaf morphism $h: \mathcal{A} \to \mathcal{B}$ $\exists$ for $\forall q \geq 0$ a group morphism $h_q: H^q(X, \mathcal{A}) \to H^q(X, \mathcal{B})$ s.t.

Functorality

$$\begin{cases} h_0 = h_X : \mathcal{A}(X) \to \mathcal{B}(X). \\ h_q = \mathrm{id} \text{ if } h = \mathrm{id} \text{ for } q \geq 0 \\ \text{given } \mathcal{A} \xrightarrow{h} \mathcal{B} \xrightarrow{g} \mathcal{C} \text{ one has } g_q \circ h_q = (g \circ h)_q. \end{cases}$$

④ For $\forall$ short exact sequence

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

$\exists$ group homomorphism $\delta^q : H^q(X, \mathcal{C}) \to H^{q+1}(X, \mathcal{A})$

s.t. the induced sequence

$$0 \to H^0(X, \mathcal{A}) \to H^0(X, \mathcal{B}) \to H^0(X, \mathcal{C}) \to H^1(X, \mathcal{A}) \to H^1(X, \mathcal{B})$$
$$\to \cdots \to H^q(X, \mathcal{A}) \to H^q(X, \mathcal{B}) \to H^q(X, \mathcal{C}) \to H^{q+1}(X, \mathcal{A}) \to \cdots$$

⑤ A commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{A} & \to & \mathcal{B} & \to & \mathcal{C} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{A}' & \to & \mathcal{B}' & \to & \mathcal{C}' & \to & 0 \end{array}$$

induces a commutative diagram

$$\begin{array}{ccccccccc} 0 \to & H^0(X, \mathcal{A}) & \to & H^0(X, \mathcal{B}) & \to & H^0(X, \mathcal{C}) & \to & H^1(X, \mathcal{A}) & \to \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & H^0(X, \mathcal{A}') & \to & H^0(X, \mathcal{B}') & \to & H^0(X, \mathcal{C}') & \to & H^1(X, \mathcal{A}') & \to \cdots \end{array}$$

★ Thm. Such cohomology theory exists & is unique.

- Soft sheaf.
  - Def. A sheaf $\mathcal{F}$ is called soft if for $\forall$ closed subset $K \subseteq X$, the restriction $\mathcal{F}(X) \to \mathcal{F}(K) = \varinjlim_{U \supset S} \mathcal{F}(U)$ is surjective.
  - e.g. The sheaf of continuous / C^1 / C^∞ functions is soft.
    But the sheaf of holomorphic functions is not soft!
  - Let $\mathcal{R}$ be a sheaf of rings & $\mathcal{M}$ a sheaf of $\mathcal{R}$-modules. Namely, each $\mathcal{M}(U)$ is an $\mathcal{R}(U)$-module.
    If $\mathcal{R}$ is soft then $\mathcal{M}$ is also soft.
  - Cor. The sheaf Ω^p of differential p-forms is soft.

• Thm. Let $0 \to \mathcal{F} \to \mathcal{F}^0 \xrightarrow{\partial^0} \mathcal{F}^1 \xrightarrow{\partial^1} \cdots$

Computation of sheaf.

be a resolution of $\mathcal{F}$ s.t. $\mathcal{F}^i$ is soft for all $i \geq 0$

Then consider the complex

$$0 \to \mathcal{F}^0 \xrightarrow{\partial^0} \mathcal{F}^1 \xrightarrow{\partial^1} \mathcal{F}^2 \xrightarrow{\partial^2} \cdots$$

One has $\ker\partial^q / \operatorname{Im}\partial^{q-1} \cong H^q(X, \mathcal{F})$. $\quad (\partial^{-1} = 0)$.

• Examples of sheaf cohomology groups.

We end this lecture by computing $H^q(M, \mathbb{R})$.

$$0 \to \mathbb{R} \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \to \cdots \quad \text{(Soft resolution)}$$

then $H^q(X, \mathbb{R}) \cong \dfrac{\ker(\Omega^q \xrightarrow{d} \Omega^{q+1})}{\operatorname{Im}(\Omega^{q-1} \xrightarrow{d} \Omega^q)}$.

On the other hand one also has

$$0 \to \mathbb{R} \to S^0 \xrightarrow{\delta} S^1 \xrightarrow{\delta} \cdots \quad \text{(this is also soft!)}$$

then one finds that

$$H^q(X, \mathbb{R}) \cong \frac{\ker(S^q \xrightarrow{\delta} S^{q+1})}{\operatorname{Im}(S^{q-1} \xrightarrow{\delta} S^q)}$$

This is exactly the de Rham theorem.

☆• Čech Cohomology. Let X be a topological space & $\mathcal{F}$ a sheaf.

Let us fix an open covering $X = \bigcup_{i \in I} U_i$, w/ I an ordered set.

Put $U_{i_0 \cdots i_p} := U_{i_0} \cap \cdots \cap U_{i_p}$ &

$$C^p(\{U_i\}, \mathcal{F}) := \prod_{i_0 < \cdots < i_p} \mathcal{F}(U_{i_0 \cdots i_p}).$$

Given $\alpha_{i_0 \cdots i_p} \in U_{i_0 \cdots i_p}$, define $\alpha_{\sigma(i_0) \cdots \sigma(i_p)} := \operatorname{Sign}(\sigma)\alpha_{i_0 \cdots i_p} \in U_{\sigma(i_0) \cdots \sigma(i_p)}$

One can define a coboundary operator $d: C^p \to C^{p+1}$ by:

for $\forall\, \alpha = \prod \alpha_{i_0 \cdots i_p} \in C^p$, define $d\alpha$ by putting

$$(d\alpha)_{i_0 \cdots i_{p+1}} := \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0 \cdots \hat{i_k}, \cdots, i_{p+1}} \Big|_{U_{i_0 \cdots i_{p+1}}}$$

e.x. Check that $d \circ d = 0$.

$\forall \alpha \in C^p(\{U_i\}, \mathcal{F})$ w/ $d\alpha$ is called a p-cochain

$\{$all the p-cochains$\} =: Z^p(\{U_i\}, \mathcal{F})$.

$\forall \beta \in C^p(\{U_i\}, \mathcal{F})$ w/ $\beta = d\alpha$ for some $\alpha \in C^{p-1}$ is p-coboundary

$\{$all p-coboundary$\} =: B^p(\{U_i, \mathcal{F}\})$.

Then define $\check{H}^p(\{U_i\}, \mathcal{F}) := \dfrac{\mathrm{Ker}(C^p \xrightarrow{d} C^{p+1})}{\mathrm{Im}(C^{p-1} \xrightarrow{d} C^p)} = \dfrac{Z^p}{B^p}$.

Note that this construction dpd on the choice of $\{U_i\}$.

If $\{V_j\}$ is a refinement of $\{U_i\}$, then there is a natural map

$$\check{H}^p(\{U_i\}, \mathcal{F}) \longrightarrow \check{H}^p(\{V_j\}, \mathcal{F}).$$

Then one defines the Čech cohomology w/o specifying an open cover as

$$\check{H}^p(X, \mathcal{F}) := \varinjlim \check{H}^p(\{U_i\}, \mathcal{F}).$$

▲ For $\forall$ open cover $\{U_i\}$, there is a natural homomorphism

$$\check{H}^p(\{U_i\}, \mathcal{F}) \to H^p(X, \mathcal{F}),$$

which is not necessarily bijective. But passing to the limit often result in isomorphisms (at least when X is paracompact)

The upshot is ① One always has $\check{H}^i(X, \mathcal{F}) \cong H^i(X, \mathcal{F})$ when $i = 0, 1$

② One has $\check{H}^i(X, \mathcal{F}) \cong H^i(X, \mathcal{F})$ for $i > 1$ when X paracpt.

So when X is a diff. mfd, one has $\check{H}^i(X, \mathcal{F}) \cong H^i(X, \mathcal{F})$, $\forall i \geq 0$.

▲ When X is a diff. mfd, one can always find a "good cover" $\{U_i\}$ s.t. $\forall$ $U_{i_0 \cdots i_p}$ is contractible. In this case,

$$\check{H}^p(\{U_i\}, \mathbb{R}) \cong H^p(X, \mathbb{R}).$$

More generally, for a sheaf $\mathcal{F}$ on X, if there is an open over $\{U_i\}$ s.t. $H^q(U_{i_0 \cdots i_p}, \mathcal{F}) = 0$ for $\forall q \geq 1$, $\forall$ $U_{i_0 \cdots i_p}$.

then one has $\check{H}^q(\{U_i\}, \mathcal{F}) \cong H^q(X, \mathcal{F})$.

Such a cover is called Leray cover, which is quite useful to compute cohomology.

- Thm. Let X be a diff. mfd w/ the constant sheaf $\mathbb{R}$. Then $\check{H}^p(X, \mathbb{R}) \cong H^p_{dR}(X, \mathbb{R})$.

pf: We will look at the case $p=2$ to illustrate the ideas. The general cases follow in a similar manner.

① We will fix a good cover $\{U_i\}$ as above. For $\forall$ representative α w/ $[\alpha] \in H^2_{dR}(X, \mathbb{R})$, since α is d-closed on each U_i, one can find a 1-form $\theta_i \in \Omega^1(U_i, \mathbb{R})$ s.t. $\alpha|_{U_i} = d\theta_i$.

Since $d(\theta_i - \theta_j) = 0$ on U_{ij}, one finds $f_{ij} \in C^\infty(U_{ij}, \mathbb{R})$ s.t. $\theta_i - \theta_j = df_{ij}$ on U_{ij}.

Note that $d(f_{ij} + f_{jk} + f_{ki}) = \theta_i - \theta_j + \theta_j - \theta_k + \theta_k - \theta_i = 0$ on U_{ijk}

So $\exists$ constant $a_{ijk} \in \mathbb{R}$ s.t. $f_{ij} + f_{jk} + f_{ki} = a_{ijk}$ on U_{ijk}.

It is easy to check that $a_{jkl} - a_{ikl} + a_{ijl} - a_{ijk} = 0$.

So $\{a_{ijk}\}$ defines a cochain in $B^2(\{U_i\}, \mathbb{R})$.

Different choices of α, θ_i, f_{ij} only differ $\{a_{ijk}\}$ by a coboundary.

So one get a map from $H^2_{dR}(X, \mathbb{R})$ to $\check{H}^2(\{U_i\}, \mathbb{R})$.

② Conversely, for $\forall$ cochain $\{a_{ijk}\}$ one can recover a 2-form α in the following magical way. Let $\{\psi_i\}$ be a partition of unity subordinate to $\{U_i\}$. Define $f_{ij} \in C^\infty(U_{ij}, \mathbb{R})$ by letting $f_{ij} := \sum_k a_{ijk}\psi_k$. Then one has

$$f_{ij} + f_{jk} + f_{ki} = \sum_l (a_{ijl} + a_{jkl} + a_{kil})\psi_l$$
$$= \sum_l a_{ijk}\psi_l = a_{ijk} \text{ on } U_{ijk}.$$

So $df_{ij} + df_{jk} + df_{ki} = 0$.

Now define $\theta_i \in \Omega^1(U_i, \mathbb{R})$ by $\theta_i := \sum_j df_{ij}\psi_j$

Then $\theta_i - \theta_j = \sum_l (df_{il} - df_{jl})\psi_l = \sum_l df_{ij}\psi_l = df_{ij}$ on U_{ij}

thus $d\theta_i - d\theta_j = 0$ on U_{ij}. So $\alpha := d\theta_i$ defines a global d-closed 2-form. So we have $\check{H}^2(X, \mathbb{R}) \to H^2_{dR}(X, \mathbb{R})$

The above two constructions are inverse to each other and generalize to $\forall p \geq 0$.

$\square$