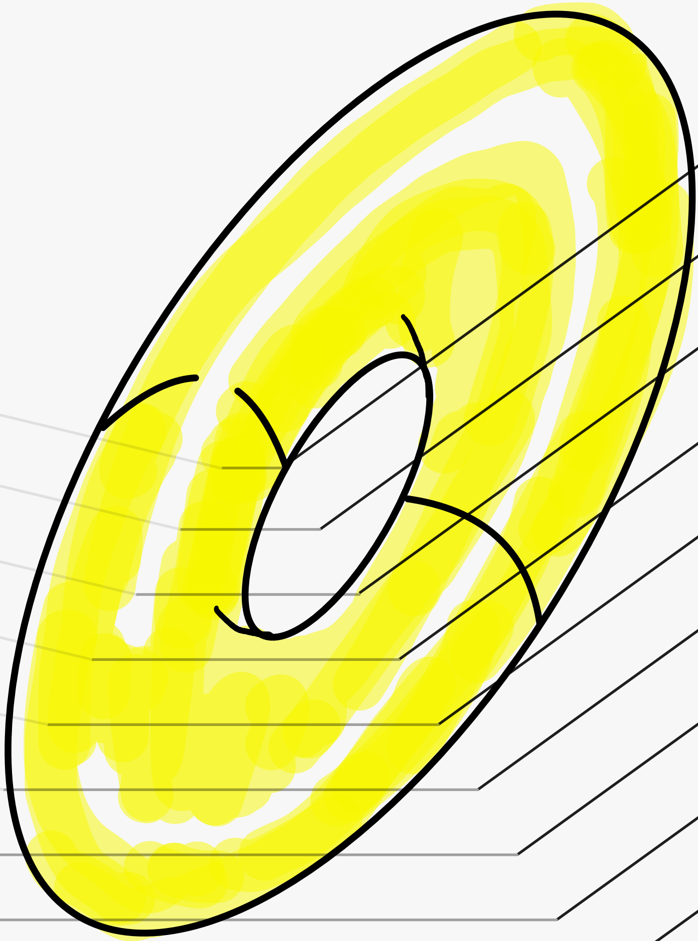
Lecture 4.

Complex Manifold.



Outline

- Definition of complex manifold
- Sheaves on a complex manifold.
- (Almost) complex structure
- (p,q)-forms
- ∂ & $\bar{\partial}$-operator

- **Def** (Complex manifold)

  Let X be a differentiable manifold. A holomorphic atlas on X is an altas $\{(U_i, \varphi_i)\}$ of the form $\varphi_i : U_i \simeq \varphi_i(U_i) \subseteq \mathbb{C}^n$ such that the transition functions

  $$\varphi_{ij} := \varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \longrightarrow \varphi_i(U_i \cap U_j)$$

  are holomorphic. The pair is (U_i, φ_i) is called a holomorphic chart. Two holomorphic atlas $\{(U_i, \varphi_i)\}$
  or a holomorphic coordinate system.
  & $\{(U_j', \varphi_j')\}$ are called equivalent if all maps

  $$\varphi_i \circ \varphi_j' : \varphi_j'(U_i \cap U_j') \longrightarrow \varphi_i(U_i \cap U_j') \text{ are holomorphic.}$$

  Such a manifold is called a complex manifold of dimension n.

  It is automatically a differentiable manifold of (real) dim $\boxed{2n}$ even dim!

  A complex mfd is called connected / compact / simply connected, etc. if the underlying differentiable mfd has this property.

- **Def.** A 1-dim cplx mfd is called a curve

  A 2-dim cplx mfd is called a surface

  A 3-dim cplx mfd is called a threefold (3-fold)

- Examples

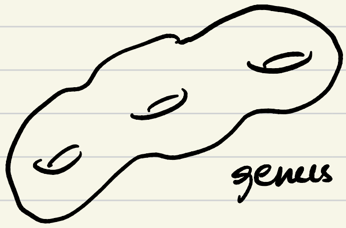
  ① $X = \mathbb{C}^n$

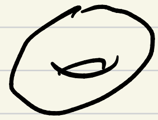
  ② $\forall$ open subset of a cplx mfd is a cplx mfd itself.

③ Riemann sphere $\mathbb{C} \cup \{+\infty\} \cong S^2 \cong \mathbb{C}P^1$.
(rational curve)

④ Riemann surface

(Algebraic curve)





elliptic curve.
$\mathbb{C}/\Gamma$ $\Gamma = z_1\mathbb{Z} + z_2\mathbb{Z}$
lattice

⑤ Complex surface.

Product : $\mathbb{C}P^1 \times \mathbb{C}P^1$
$S^2 \times S^2$

Hopf surface $S^1 \times S^3 \cong \mathbb{C}^2 \setminus \{0\} / \mathbb{Z}$

$\mathbb{Z}$ action is defined by $(z_1, z_2) \overset{k\cdot}{\longmapsto} \left(\left(\tfrac{1}{2}\right)^k z_1, \left(\tfrac{1}{2}\right)^k z_2\right)$

⑥ Complex projective space $\mathbb{C}P^n = \mathbb{C}^{n+1} \setminus \{0\} / \sim$.

$U_i = \left(\mathbb{C}^n, \frac{z_1}{z_i}, \cdots, \widehat{\frac{z_i}{z_i}}, \cdots, \frac{z_n}{z_i}\right)$

⑦ Grassmannian mfd. Let V be a complex vector space of dim $n+1$. Then the set

$$Gr_k(V) := \{ W \subseteq V \mid \dim(W) = k \}$$

can be endowed w/ a structure of cplx mfd s.t. $\dim Gr_k(V) = k(n+1-k)$

• Def. Let X be a cplx mfd of dim n. Let $Y \subseteq X$ be a differentiable submfd of real dim $2k$. Then Y is called a cplx submfd if $\exists$ holomorphic atlas $\{(U_i, \varphi_i)\}$ of X such that

$$\varphi_i : U_i \cap Y \xrightarrow{\cong} \varphi_i(U_i) \cap \mathbb{C}^k,$$

where $\mathbb{C}^k$ is embedded in $\mathbb{C}^n$ via $(z_1, \cdots, z_k) \to (z_1, \cdots, z_k, 0, \cdots, 0)$

Namely, locally $\exists$ holo. coord. $(z_1, \cdots, z_n)$ s.t.
Y is cut out by $z_{k+1} = \cdots = z_n = 0$.

- A complex mfd X is called projective if it is isomorphic to a closed complex submfd of some complex projective space $\mathbb{C}P^N$.

- **Def.** A function $f: X \to \mathbb{C}$ on a cplx mfd is called holomorphic if $f \circ \varphi_i^{-1}: \varphi_i(U_i) \to \mathbb{C}$ is a holomorphic function for $\forall$ chart (U_i, φ_i).

- **Thm.** A holomorphic function on a cpt cplx mfd must be a constant function.

pf: By max principle we know that $f \equiv C_0$ for some $C_0 \in \mathbb{C}$ around the max point of $|f|$. Then let
$\Omega := \{x \mid D^k f(x) = 0 \text{ for } \forall k \geq 1\}$. Then clearly Ω is closed & non-empty. Ω is also open by power series expansion.
So $\Omega = X$ as X is connected.
$\square$.

- There is no compact complex submfd in $\mathbb{C}^N$ with positive dimension.

pf: Let $X \subseteq \mathbb{C}^N$ be a cpt cplx mfd.
Then the restrictions of $z_1, \cdots, z_N$ on X must be constants so in particular X is a pt. $\square$.

- Some sheaves on cplx mfd.

1. $\mathcal{O}_X$ Structure sheaf
$$\mathcal{O}_X(U) = \{ f : U \to \mathbb{C} \text{ holomorphic} \}.$$

2. $\mathcal{M}$ sheaf of meromorphic functions.
$$\mathcal{M}(U) = \{ f : U \text{ meromorphic} \}$$
$\exists$ open cover $\{U_i\}$ of U s.t. $f|_{U_i} = \frac{g_i}{h_i}$ for some $g_i, h_i \in \mathcal{O}_X(U_i)$.

3. $\mathcal{O}_X^*$ sheaf of holomorphic functions vanishing nowhere.

4. $Y \subseteq X$ cplx submanifold.

$\mathcal{I}_Y$: ideal sheaf of Y.
$$\mathcal{I}_Y(U) = \{ f \in \mathcal{O}(U) \mid f \equiv 0 \text{ on } U \cap Y \}.$$
We have an exact sequence
$$0 \to \mathcal{I}_Y \to \mathcal{O}_X \to \mathcal{O}_Y \to 0.$$

- If X is cpt cplx mfd, then
$$\Gamma(X, \mathcal{O}_X) = H^0(X, \mathcal{O}_X) = \mathbb{C}.$$

- However, $\Gamma(X, \mathcal{M}) =: K(X)$, the spaces of meromorphic functions on X could be really large. In fact $K(X)$ is a field extension over $\mathbb{C}$. the transcendence degree $\operatorname{trdeg}_{\mathbb{C}} K(X)$ is called the algebraic dimension of X.

- (Non-trivial fact) One has $\operatorname{trdeg}_{\mathbb{C}} K(X) \leq \dim X$.
Also note that it's possible that $\operatorname{trdeg}_{\mathbb{C}} K(X) = 0$, namely there is no non-trivial meromorphic function.
e.g. X surface of class VII.

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When $\mathrm{trdeg}_{\mathbb{C}} K(X) = \dim X$, X is called Moishezon which is equivalently to saying that X is bimeromorphic to a projective mfd. (iff X carries a big line bundle)

In fact after a sequence of blowups w/ smooth centers X can become a smooth proj. mfd.)

X is Moishezon iff it carries an integral Kähler current.

A Kähler mfd is Moishezon iff it is projective.

- The biggest difference between differentiable mfd & cplx mfd is the complex structure, which plays the role of "$\sqrt{-1}$".

- For a cplx vector space V of dim n, one can think of it as a real vector space of dim $2n$. Namely, if $\{e_1, \cdots, e_n\}$ is a $\mathbb{C}$-basis of V then $\{e_1, \cdots, e_n, \sqrt{-1}e_1, \cdots, \sqrt{-1}e_n\}$ is an $\mathbb{R}$-basis of V.
  One can then think of "$\sqrt{-1}$" as an $\mathbb{R}$-linear transformation.
  And we have $\sqrt{-1} \circ \sqrt{-1} = -\mathrm{id}$.

- If V is a real vector space w/ an $\mathbb{R}$-linear transformation $J: V \to V$ satisfying $J^2 = -\mathrm{id}$. Then J is called a cplx structure of V. We can then equip V w/ a structure of cplx vector space by putting
$$(a + \sqrt{-1}b)\cdot v := av + bJv \quad \text{for } \forall\, a+\sqrt{-1}b \in \mathbb{C}.$$
  Note that V must have even real dimension.
  In fact, let $\{e_1, \cdots, e_n\}$ be a $\mathbb{C}$-basis of the above cplx space then $\{e_1, \cdots, e_n, Je_1, \cdots, Je_n\}$ is an $\mathbb{R}$-basis of the original real vector space. So in particular it's even dimensional.

- If V is a real vector space then we can make it complex by putting $V^{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$. Namely if $\{e_1, \dots, e_k\}$ is an $\mathbb{R}$-basis of V then $\{e_1, \dots, e_k, \sqrt{-1}e_1, \dots, \sqrt{-1}e_k\}$ is a $\mathbb{C}$-basis for $V^{\mathbb{C}}$

- Now let V be a real vector space of dim $2n$ w/ a cplx structure J. Then one has

$$\boxed{V^{\mathbb{C}} = V^{1,0} \oplus V^{0,1}}$$

where

$$V^{1,0} = \{ v \in V^{\mathbb{C}} \text{ s.t. } Jv = \sqrt{-1}v \}$$
$$V^{0,1} = \{ v \in V^{\mathbb{C}} \text{ s.t. } Jv = -\sqrt{-1}v \}.$$

If $\{e_1, \dots, e_n, Je_1, \dots, Je_n\}$ is an $\mathbb{R}$-basis of V then

$$\begin{cases} \{e_i - \sqrt{-1}Je_i\}_{i=1}^n \text{ is a } \mathbb{C} \text{ basis of } V^{1,0} \\ \{e_i + \sqrt{-1}Je_i\}_{i=1}^n \text{ is a } \mathbb{C}\text{-basis of } V^{0,1}. \end{cases}$$

More generally for the p-th wedge $\Lambda^p V^{\mathbb{C}}$ one has

$$\Lambda^p V^{\mathbb{C}} = \bigoplus_{r+s=p} V^{r,s}$$

where $V^{r,s}$ is generated by elements of the form $\omega \wedge v$ w/ $\omega \in \Lambda^r V^{1,0}$ & $v \in \Lambda^s V^{0,1}$.

Example

- Applying the above argument to differential forms on $\mathbb{C}^n$, we let $V = \mathrm{Span}_{\mathbb{R}}\{dx^1, \cdots, dx^n, dy^1, \cdots, dy^n\}$

  then $V^{\mathbb{C}} = \mathrm{Span}_{\mathbb{C}}\{dx^1, \cdots, dx^n, dy^1, \cdots, dy^n\}$ where

  J is defined by $\begin{cases} J\,dx^i = -dy^i \\ J\,dy^i = dx^i \end{cases}$.

  Then $V^{1,0} = \mathrm{Span}_{\mathbb{C}}\{dx^i + \sqrt{-1}\,dy^i\} = \mathrm{Span}_{\mathbb{C}}\{dz^1, \cdots, dz^n\}$

  & $V^{0,1} = \mathrm{Span}_{\mathbb{C}}\{dx^i - \sqrt{-1}\,dy^i\} = \mathrm{Span}_{\mathbb{C}}\{d\bar{z}^1, \cdots, d\bar{z}^n\}$

  So in particular $\forall$ complex 1-form $\alpha = a_i dx^i + b_j dy^j$ can be uniquely decomposed as $\alpha = \alpha^{1,0} + \alpha^{0,1}$ w/ $\alpha^{1,0} = f_i dz^i + g_j d\bar{z}^j$ for some f_i, g_j complex valued function

  More generally, $\forall$ complex p-form α can be decomposed by $\alpha = \sum_{r+s=p} \alpha^{r,s}$ w/ $\alpha^{r,s} = \sum a_{I,\bar{J}}\, dz^I \wedge d\bar{z}^{\bar{J}}$.

  $I = (i_1, \cdots, i_r)$

  $J = (j_1, \cdots, j_s)$.

- Let X be a differentiable mfd of dim $2n$.

  Assume that there is a morphism $J : TX \to TX$ satisfying $J^2 = -id$. Then J is called an **almost complex structure** of X. Rmk. Such X must be orientable. prove this using

  This J also induces a morphism on T^*X.

  So $\forall$ complex valued p-form α on X can be decomposed (w.r.t. J) into $\alpha = \sum_{r+s=p} \alpha^{r,s}$.

  In other words

$$\Omega^p = \bigoplus_{r+s=p} \Omega^{r,s}_J$$

Here Ω^p denote the sheaf of complex valued p-forms on X & $\Omega^{r,s}_J$ is the sheaf of (r,s) forms w.r.t. J.

- We define the operator ∂ & $\bar\partial$ (again w.r.t. J) by letting
$$\begin{cases} \partial := (d)^{r+1,s} : \Omega^{r,s}_J \to \Omega^{r+1,s}_J \\ \bar\partial := (d)^{r,s+1} : \Omega^{r,s}_J \to \Omega^{r,s+1}_J . \end{cases}$$
This is defined for $\forall\, r,s \geq 0$.

- Def. We say J is integrable if $d = \partial + \bar\partial$.

- $\forall$ complex mfd has an integrable complex structure given as follows: locally $(z^1, \cdots, z^n)$, using the complex structure in the previous example.
$z^1 = x^1+\sqrt{-1}y^1$, $z^n = x^n+\sqrt{-1}y^n$

So $J = -\frac{\partial}{\partial x^i} \otimes dy^i + \frac{\partial}{\partial y^j} \otimes dx^j$

This gives rise to a global section of $T^*M \otimes TM$

e.x. Check that J such defined is integrable.

Namely, for $\forall$ (r,s) form α, $d\alpha = \alpha^{r+1,s} + \alpha^{r,s+1}$.

- Conversely (Deep result due to Newlander-Nirenberg)

An integrable cplx structure J determines a unique structure of complex mfd for X whose associated cplx structure is J itself.

- J is integrable iff the Nijenhuis tensor
$$N_J(U,V) := [U,V] + J([JU,V]+[U,JV]) - [JU,JV]$$
vanishes identically.

- Big conjecture. Is there an integrable complex structure on S^6?

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- Let X be a cpt cplx surface (so $\dim X = 2$)

Assume that $\alpha \in H^0(X, \Omega_X^{1,0})$ is a holomorphic $(1,0)$ form.

i.e. $\bar\partial \alpha = 0$

Then $d\alpha = 0$.

pf: Observe that $d\alpha$ is a holomorphic $(2,0)$ form ($\bar\partial\, d\alpha = 0$)

Consider $d\alpha \wedge \bar\alpha$, which is $(2,1)$ form.

So Stokes's formula shows that

$$0 = \int_X d(d\alpha\wedge\bar\alpha) = \int_X \bar\partial(d\alpha\wedge\bar\alpha) = -\int_X d\alpha \wedge \overline{\partial\alpha}.$$

Using $\partial\alpha = d\alpha$, we find that $\int_X d\alpha\wedge\overline{d\alpha} = 0$.

Then $d\alpha = 0$, as desired.

e.x. show non-negative volume form