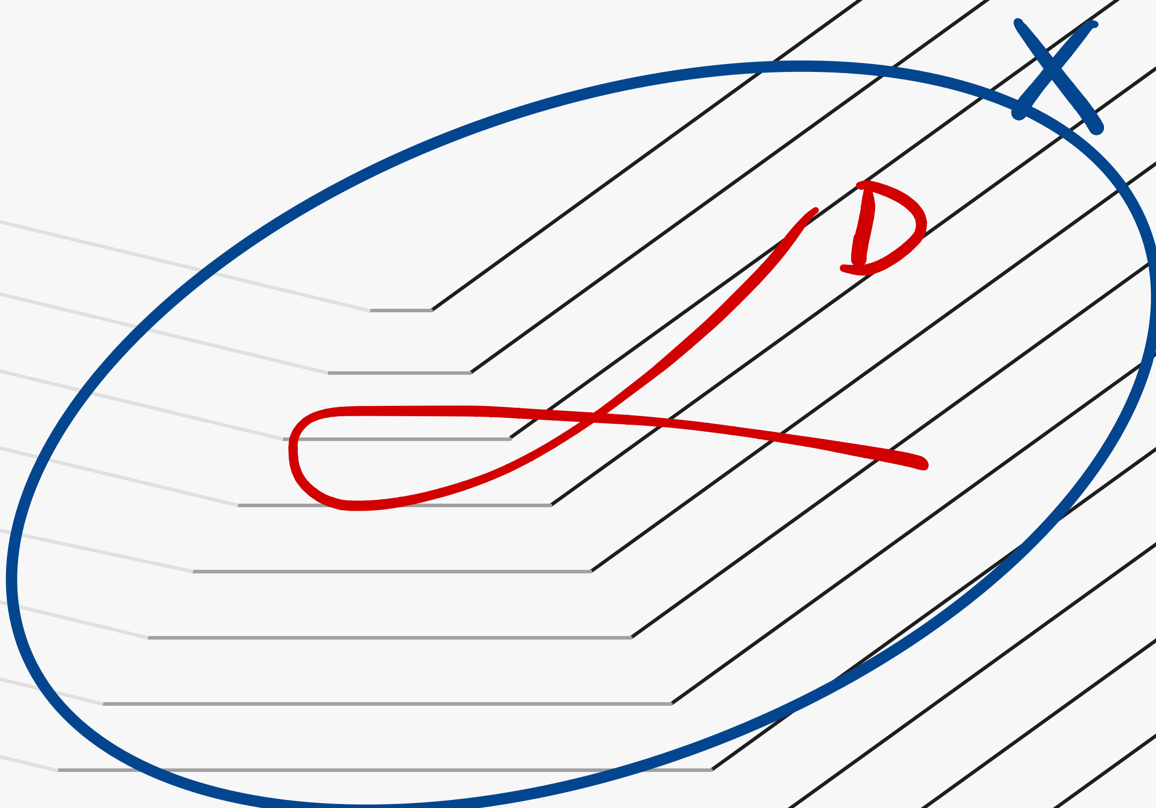
Lecture 6

Holomorphic line bundle

Part I & Part II.



Outline

- Definition & examples
- Picard group
- First chern class
- Divisors & global sections
- Volume of line bundles
- Section ring
- Normal bundle & adjunction formula.
- Hermitian innerproduct & Chern curvature

- <u>Def.</u> A holomorphic line bundle is a holomorphic vector bundle of rank 1.

  Equivalently, $\exists$ local cover $\{U_i\}$ of X & $\exists$ holomorphic function $f_{ij} \in \mathcal{O}^*(U_i \cap U_j)$ for $\forall i,j$ s.t.

  $$f_{ij} \cdot f_{jk} = f_{ik}. \quad \& \quad f_{ii} = 1.$$

- ▲ The above data gives rise to a holo. line bundle L by letting

  $$L = \coprod_i U_i \times \mathbb{C} \Big/ \sim \quad w/ \quad (x,z) \in U_i \times \mathbb{C} \ \& \ (y,w) \in U_j \times \mathbb{C}$$

  satisfying $(x,z) \sim (y,w)$ iff $x = y$ & $z = f_{ij} w$.

- Let L & L' be two holo. line bundle. Then one can always find a common trivialization $\{U_i\}$ s.t. $L|_{U_i}$ & $L'|_{U_i}$ are both trivialized for all i. Let $\{f_{ij}\}$ & $\{g_{ij}\}$ be the transition functions of L & L' respectively. Then we say

  $\boxed{L \ \& \ L' \text{ are isomorphic}}$ if one can find $h_i \in \mathcal{O}^*(U_i)$ for $\forall i$

  s.t. $\frac{h_i}{h_j} f_{ij} = g_{ij}$ on $U_i \cap U_j$ for $\forall i,j$.

  This is equivalent to saying that there exists a bundle homomorphism $\varphi : L \to L'$ which gives rise to a bundle isomorphism.

• Thm. Isomorphism classes of holo. line bundles are in 1 to 1 correspondence to elements in $H^1(X, \mathcal{O}_X^*)$.

Pf: Note that $H^1(X, \mathcal{O}_X^*) \cong \check{H}^1(X, \mathcal{O}_X^*)$.

For $\forall$ two isomorphic line bundles L & L', consider their common trivialization $\mathcal{U} = \{U_i\}$. Then their transition functions give rise to two elements in $Z^1(\mathcal{U}, \mathcal{O}^*)$, say α & α'. More specifically, $\begin{cases} \alpha = \{U_i \cap U_j, g_{ij}\} \\ \alpha' = \{U_i \cap U_j, g'_{ij}\} \end{cases}$ w/ g_{ij} & g'_{ij} satisfying the cocycle relation.

Since L & L' are isomorphic, $\exists\ f_i \in \mathcal{O}_X^*(U_i)$ s.t. $\frac{f_i}{f_j} = \frac{g_{ij}}{g'_{ij}}$ so that $\alpha - \alpha' \in B^1(\mathcal{U}, \mathcal{O}^*)$.

Thus α & α' correspond to the same element in $\check{H}^1(\mathcal{U}, \mathcal{O}^*)$.

So α & α' gives the same element in $\check{H}^1(X, \mathcal{O}^*)$.

This element doesn't dpd on the choice of the covering $\mathcal{U}$. Indeed, given another covering $\mathcal{V}$, one can find a common refinement $\mathcal{W}$ of $\mathcal{U}$ & $\mathcal{V}$ s.t. L & L' give the same element in $\check{H}(\mathcal{W}, \mathcal{O}^*)$ (e.x. Check this). So taking direct limit yields a well-defined element in $\check{H}^1(X, \mathcal{O}^*)$.

Now conversely, $\forall$ element in $\check{H}(X, \mathcal{O}^*)$ can be realized as an open cover $\{U_i\}$ together w/ $g_{ij} \in \mathcal{O}^*(U_i \cap U_j)$ s.t. g_{ij} satisfies cocycle condition, which corresponds to a holo. line bundle. This bundle is uniquely determined up to isomorphism. $\square$.

• Prop: Let $\mathcal{O}_X$ denote the trivial holo. line bundle, i.e. $\mathcal{O}_X = X \times \mathbb{C}$. Then for $\forall$ holo. line bundle L, one has

Justify this notation

① $L \otimes \mathcal{O}_X \cong \mathcal{O}_X \otimes L \cong L$.

This is only defined up to isomorphism.

② $L \otimes L^* \cong \mathcal{O}_X$.

③ $L_1 \otimes L_2 \cong L_2 \otimes L_1$.

pf: Prove this using transition functions. $\square$.

- **Prop.** The tensor product & dual endow the set of all isomorphism classes of holo. line bundles an abelian group struct which we denote by $\mathrm{Pic}(X)$ — the Picard group of X.
  There is a natural isomorphism $\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^*)$.

pf: The isomorphism is given by the description of holo. l.b. using cocycle transition functions. $\square$.

- Consider the exact sequence of sheaves:

$$0 \to \mathbb{Z} \xrightarrow{i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0.$$

then one has an induced exact sequence

$$\begin{tikzcd} \to H^1(X,\mathcal{O}_X) \arrow[r] & H^1(X, \mathcal{O}_X^*) \arrow[r, "\delta"] \arrow[d, phantom, "\text{\rotatebox{90}{$\cong$}}"] & H^2(X, \mathbb{Z}) \arrow[r] & H^2(X,\mathcal{O}_X) \to \\ & \mathrm{Pic}(X) \arrow[ur, dashed, "c_1"'] & & \end{tikzcd}$$

The induced map $\mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$ is denoted by $\boxed{c_1}$. For $\forall\, L \in \mathrm{Pic}(X)$, $c_1(L)$ is called the first Chern class of L.

- If $L \cong \mathcal{O}_X$, i.e., L trivial, then $c_1(L) = 0$.

pf: This is direct from the above definition. $\square$.

- However conversely, if $c_1(L)=0$, then L is not necessarily trivial. Such a line bundle must come from the image of $H^1(X,\mathcal{O}_X)\to H^1(X,\mathcal{O}_X^*)$, which is identified w/ the kernel of c_1
  it is indeed trivial as a cplx line bundle. Picard variety (Abelian Variety)
- We will give a different description of $c_1(L)$ later.

- **Def.** Let E be a holo. vector bundle. Then the first Chern class $c_1(E)$ is defined to be $c_1(\det E)$.
- **Def.** The first chern class of a cplx mfd is defined to be $c_1(X):=c_1(TX^{1,0})$.
- $\boxed{\text{Additive Convention}}$ Since $\mathrm{Pic}(X)$ is Abelian, one often uses additive convention:
$$\begin{cases} L_1+L_2 := L_1\otimes L_2 \\ -L := L^* \end{cases}$$
Using this convention, one has $\begin{cases} c_1(X)=c_1(-K_X). \\ c_1(L_1+L_2)=c_1(L_1)+c_1(L_2). \end{cases}$

- **Example.**
  nontrivial fact ★ $\forall$ holomorphic line bundle L over $\mathbb{CP}^n$ is isomorphic to $\mathcal{O}(k)$ for some $k\in\mathbb{Z}$.
  So the Picard group of $\mathbb{CP}^n$ is isomorphic to $\mathbb{Z}$.

  → This can be proved using the fact $H^i(X,\mathcal{O}_X)=0$, $i>0$ and the exponential sequence.

- Given a general cplx line bundle L over a diff. mfd X, one can also define $c_1(L) \in H^2(X, L)$. There are several different ways to describe $c_1(L)$ (Using Čech cohomo, connection, topological constru. ⋯)
We will use Čech cohomology. Let us choose a sufficiently "fine" covering $\mathcal{U} = \{U_i\}$ s.t. on $U_i \cap U_j$, the transition function g_{ij} of L can be written as $g_{ij} = e^{2\pi i f_{ij}}$ for some $f_{ij} \in C^\infty(U_{ij}, \mathbb{C})$. Here f_{ij} is defined up to an integer.
Now the cocycle condition $g_{ij} g_{jk} g_{ki} = 0$ implies that $f_{ij} + f_{jk} + f_{ki} = a_{ijk} \in \mathbb{Z}$. So $\{U_i \cap U_j \cap U_k, a_{ijk}\}$ defines a cochain in $\check{H}^2(\{U_i\}, \mathbb{Z}) \cong H^2(X, \mathbb{Z})$.

- But the above description is sometimes not adequate for calculation. In the view of $\check{H}^2(X, \mathbb{R}) \cong H^2_{dR}(X, \mathbb{R})$ it is more useful to construct d-closed 2 forms from a cplx line bundle. This is done using connections & curvatures. We will not go in this direction in this course.

- The above construction of $c_1(L)$ for cplx line bundles can also be described using cohomology.
Let A_X be the sheaf of $\mathbb{C}$-valued smooth functions on X.
Let A_X^* denote the sheaf of $\mathbb{C}$-valued invertible C^∞ functions on X.
Then similarly as in the case of holo. l.b., isomorphism classes of cplx l.b. are in 1-1 corresp. w/ $H^1(X, A_X^*)$
Then the exact sequence $0 \to \mathbb{Z} \xrightarrow{i} A_X \xrightarrow{\exp(2\pi i f)} A_X^* \to 0$ induces a map $H^1(X, A_X^*) \xrightarrow{c_1} H^2(X, \mathbb{Z})$ which yields the first Chern class $c_1(L)$ for $\forall$ cplx. l.b. L over differentiable mfd.

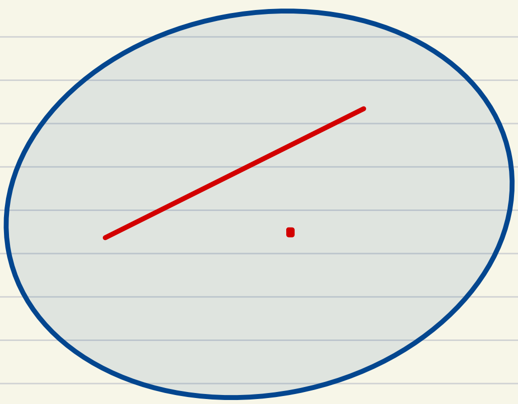
- **Def** $\boxed{\text{Analytic subvariety.}}$ An analytic subvariety is a closed subset $Y \subseteq X$ s.t. for $\forall\, x \in X$ $\exists$ open nbhd $x \in U \subseteq X$ s.t. $Y \cap U$ is the zero set of finitely many holomorphic functions $f_1 \cdots f_k \in \mathcal{O}(U)$.

  A pt $y \in Y$ is called smooth if $\exists$ nbhd $y \in U \subseteq X$ s.t. $U \cap Y$ is a complex submfd. Namely $\exists$ local holo. coord. $(z^1, \cdots, z^n)$ around y s.t. $U \cap Y = \{ z^1 = \cdots = z^k = 0 \}$. In this case we say Y has codim k at y.

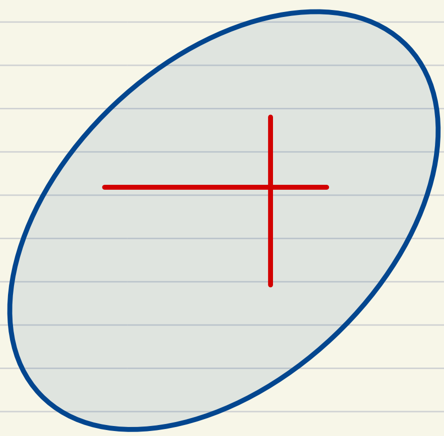
  ▲ Note that codim is locally const. on smooth locus.

  We denote all the regular pt of Y be Y_{reg}. Let $Y_{sing} = Y \setminus Y_{reg}$.
  Then Y_{reg} is a cplx submfd of X.

  Y is called irreducible if Y cannot be written as $Y = Y_1 \cup Y_2$ for proper analytic subvariety $Y_i \subseteq Y$.



Y = line $\cup$ point



Y = Union of two lines

Note that $\begin{cases} Y_{sing} \subseteq Y \text{ is also an analytic subvariety of } X. \\ Y_{reg} \text{ is open dense connected in } Y. \end{cases}$

The dimension of an irreducible variety Y is defined by

$$\dim Y := \dim Y_{reg} \qquad \operatorname{codim} Y = \dim X - \dim Y.$$

- **Def.** An irreducible analytic subvariety is called a $\boxed{\text{prime divisor.}}$ (or irred. hypersurface).

- **Def.** A $\boxed{\text{(Weil) divisor}}$ on X is a formal linear combination

$$D = \sum a_i Y_i, \quad a_i \in \mathbb{Z}, \ Y_i \text{ prime divisor.}$$

▲ We require the above sum to be locally finite.
Namely for $\forall x \in X$ $\exists$ nbhd $x \in U \subseteq X$ s.t. $\#\{a_i \neq 0 \mid Y_i \cap U \neq \phi\} < +\infty$.
If X is cpt, this amounts to saying that $\sum a_i Y_i$ is a finite sum.

- A divisor $D = \sum a_i Y_i$ is said to be $\boxed{\text{effective}}$ if $a_i \geq 0$ for $\forall i$.
  In this case we write $D \geq 0$.

- $\forall$ divisor is the difference of two effective divisors.

- Def. let $f \in K(X)$ be a non-zero global meromorphic function.
  One can define $\mathrm{ord}_Y(f)$ for $\forall$ prime divisor Y as follows:
  pick a regular pt $y \in Y_{reg}$. Then locally Y is cut out by an irreducible element $g \in \mathcal{O}_{X,y}$. Also locally $f = \frac{h}{l}$ for $h, l \in \mathcal{O}_{X,y}$.
  Then $\mathrm{ord}_Y(f) := \mathrm{ord}_Y(h) - \mathrm{ord}_Y(l)$ w/ $\mathrm{ord}_Y(h)$ given by $f = g^{\mathrm{ord}_Y(h)} t$, $t \in \mathcal{O}^*_{X,y}$.
  this definition is indpd of the choice of y (as $\mathrm{ord}_y(f)$ is locally const. & Y_{reg} is connected).

- For $f \in K(X)$, $f \neq 0$, we let
  $$\mathrm{div}(f) := \sum_{Y \text{ prime}} \mathrm{ord}_Y(f)\, Y.$$
  Such a divisor is called a $\boxed{\text{principle divisor}}$.

▲ $\mathrm{div}(f_1 f_2) = \mathrm{div}(f_1) + \mathrm{div}(f_2)$.
$\mathrm{div}(f_1 + f_2) \geq \mathrm{div} f_i$, $i = 1, 2$. ← the difference is effective.

- Example of principle divisors
  Let L be a holomorphic line bundle.
  Assume that $H^0(X, L) \neq 0$. Then $\forall$ $0 \neq s \in H^0(X, L)$ cuts out a divisor $(s=0) := \sum_Y \mathrm{ord}_Y(s) Y$. If $(s=0) = 0$ then $L \simeq \mathcal{O}_X$.
  e.x. Show that $\mathrm{ord}_Y(s)$ is well-defined.
  For $\forall$ two $s_1, s_2 \in H^0(X, L)$, $(s_1=0) - (s_2=0)$ is principle, as s_1/s_2 is a globally defined meromorphic function.

- Let $WDiv(X)$ be the abelian group generated by Weil divisors. Then one can identify $WDiv(X)$ w/ $H^0(X, K_X^*/\mathcal{O}_X^*)$.

In algebraic geometry, $H^0(X, K^*/\mathcal{O}^*)$ is called the group of Cartier divisors. On cplx mfd these two notions coincide, but for general analytic varieties they may differ.

Pf: Given $D \in WDiv(X)$, locally, $D = (\prod f_i^{a_i} = 0)$ $f_i \in \mathcal{O}$, $a_i \in \mathbb{Z}$.

On overlaps $\prod f_i^{a_i}$ differ by an element in $\mathcal{O}_X^*$.

Conversely, $\forall$ element in $H^0(K^*/\mathcal{O}^*)$ locally given by $h_i \in K^*(U_i)$ w/ $h_i/h_j \in \mathcal{O}^*(U_i \cap U_j)$. So $\mathrm{ord}_Y(h_i) = \mathrm{ord}_Y(h_j)$ for $\forall i, j$. $\forall Y$ prime

Thus we get a weil divisor. $\square$.

- So we will simply denote $Div(X) = WDiv(X) = H^0(X, K_X^*/\mathcal{O}_X^*)$

$\forall D \in Div(X)$ gives rise to a holomorphic line bundle. Indeed, think of D as a Cartier divisor, one use $\{U_i, h_i\}$ $h_i \in K^*(U_i)$ to define $g_{ij} := h_i/h_j$ as the transition functions, whose resulting holo. line bundle will be denoted by $\mathcal{O}(D)$.

▲ e.x. Show that $\mathcal{O}(D_1 + D_2) \cong \mathcal{O}(D_1) \otimes \mathcal{O}(D_2)$

- In general it is not true that $\forall$ holo. line bundle L is of the form $L \cong \mathcal{O}(D)$ for some $D \in Div(X)$

▲ e.x. $L \cong \mathcal{O}(D)$ iff L has a global meromorphic section.

▲ Rmk. When X is proj. $\forall L$ is of the form $L = \mathcal{O}(D)$ for some D.

Exam. $\mathcal{O}(D) \cong \mathcal{O}_X$ iff D is principle divisor.

- If $D \geq 0$. Then locally on $\{U_i\}$ one can find $f_i \in \mathcal{O}(U_i)$ s.t. $D = \mathrm{div}(f_i)$. In this case, one sees that (U_i, f_i) itself gives rise to a global holomorphic section in $H^0(X, \mathcal{O}(D))$, which is denoted by S_D. It is called the defining section of D.

- **Prop.** $H^0(X, \mathcal{O}(D)) \cong \{ f \in K^*(X) \mid \mathrm{div}(f) + D \geq 0 \} \cup \{0\}$.

  $0 \longmapsto f \equiv 0$ (zero section)

  pf: Assume that $D = \{U_i, h_i\}$, $h_i \in K^*(U_i)$.
  Then for $\forall\, 0 \neq s \in H^0(X, \mathcal{O}(D))$, $s = \{U_i, f_i\}$, $f_i \in \mathcal{O}(U_i)$
  Note that $f_i/h_i = f_j/h_j$. So it defines an element say f in $K^*(X)$. Then $\mathrm{div}(f) + D|_{U_i} = \mathrm{div}(f_i) \geq 0$.
  Conversely, given $\forall f \in K^*(X)$ w/ $\mathrm{div}(f) + D \geq 0$, may define $f_i := f h_i \in \mathcal{O}(U_i)$. Then $\{U_i, f_i\}$ defines an element in $H^0(X, \mathcal{O}(D))$
  The above correspondence is linear & bijective. $\square$

- We end this lecture by showing that $c(L)$ completely determines L as cplx line bundle (namely we forget the holo. stru.)
  Let A_X denote the sheaf of cplx valued smooth functions.
  Then we have an inclusion $\mathcal{O}_X \hookrightarrow A_X$ & $\mathcal{O}_X^* \hookrightarrow A_X^*$.
  Note that, $\forall$ isomorphism class of cplx line bundle is in 1-1 corresp. w/ element in $H^1(X, A_X^*)$. Consider exact sequences:

$$
\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "i"] \arrow[d, equal] & \mathcal{O}_X \arrow[r, "\exp(2\pi i\,\cdot)"] \arrow[d] & \mathcal{O}_X^* \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "i"] & A_X \arrow[r, "\exp(2\pi i\,\cdot)"] & A_X^* \arrow[r] & 0
\end{tikzcd}
$$

This induces

$$\begin{tikzcd}
\to H^1(X,\mathcal{O}_X) \arrow[r] \arrow[d] & H^1(X,\mathcal{O}_X^*) \arrow[r] \arrow[d, "f\text{: forget the holo. structure}"] & H^2(X,\mathbb{Z}) \arrow[r] \arrow[d, equal] & H^2(X,\mathcal{O}_X) \to \cdots \arrow[d] \\
\to H^1(X,\mathcal{A}_X) \arrow[r] & H^1(X,\mathcal{A}_X^*) \arrow[r] & H^2(X,\mathbb{Z}) \arrow[r] & H^2(X,\mathcal{A}_X) \to \cdots
\end{tikzcd}$$

$H^1(X,\mathcal{A}_X) = 0$, $H^2(X,\mathcal{A}_X) = 0$. $\mathcal{A}_X$ is soft.

So $H^1(X,\mathcal{A}_X^*) \cong H^2(X,\mathbb{Z})$. $\forall$ element in $H^2(X,\mathbb{Z})$ determines a cplx line bundle on X.

Moreover, $\forall$ holo. line bundle w/ trivial c_1 is trivial as a cplx line bundle. Namely locally can find $f_i \in \mathcal{A}^*(U_i)$ s.t. $g_{ij} = \frac{f_i}{f_j}$.

Part II

- Let L be a holo. line bundle. Then $H^0(X,L)$ can be treated as generalized holo. functions on X. They encode rich information!

Thm

- If X is cpt, then for $\forall$ holo. line bundle L, $H^0(X,L)$ is finite dimension.

Pf: Define a norm $\|\cdot\|$ on $H^0(X,L)$ by letting

$$\|s\|^2 := \int_X h(s,s)\, dV,$$

where h is a smooth Hermitian metric on L & dV is any smooth volume form.

Then we need to show that the unit sphere

$$B := \{ s \in H^0(X,L) \mid \|s\| = 1 \}$$ is compact. Riesz lemma.

This follows from the fact that $\forall$ higher order derivatives of a holo. function can be uniformly controlled using L^2-norm. Then cptness follows from Weierstrass convergence. $\square$

- Let X be n-dim cpt cplx mfd w/ a holo. line bundle L.
  We define the volume of L to be
  $$\mathrm{Vol}(L) := \limsup_{m\to\infty} \frac{\dim_{\mathbb{C}} H^0(X, mL)}{m^n/n!}$$
  ($mL = L^{\otimes m}$)
  We say L is big if $\mathrm{Vol}(L) > 0$. In this case X is Moishezon.
  (So X is bimeromorphic to a proj. mfd).

- Let L be a holo. line bundle over a cplx mfd X.
  Put $R(X,L) := \bigoplus_{m\geq 0} H^0(X, mL)$. Here $H^0(X, 0\cdot L) := H^0(X, \mathcal{O}_X)$.
  Then $R(X,L)$ is called the section ring of L.
  Why is there a ring structure?
  For $\forall\, s_1 \in H^0(X, m_1 L)$ & $s_2 \in H^0(X, m_2 L)$, one can define $s_1 \otimes s_2 \in H^0(X, (m_1+m_2)L)$ using local data of L.
  e.x. Check this.
  When $L = K_X$, the ring $R(X, K_X)$ is called the $\boxed{\text{canonical ring}}$ of X. Many properties of X is determined by the canonical ring.

- Rmk. It is possible that $H^0(X, mL) = 0$ for $\forall\, m > 0$. When $-L$ ample.
  It is also possible that $H^0(X, mL) \equiv 1$ for $\forall\, m \geq 0$. When $L = \mathcal{O}(E)$, E exceptional

- Example. $X = \mathbb{CP}^n$, $L = \mathcal{O}(1)$.
  Then $H^0(X, mL) = \{$homogeneous polynomials $f(z_0, \cdots, z_n)$ of degree $m\}$.
  And $R(X,L) =$ the ring of polynomials in $(n+1)$-variables.

- Let $Y \subseteq X$ be a cplx submfd. Then there is an exact sequence of sheaves:

$$0 \to \mathcal{I}_Y \to \mathcal{O}_X \to \mathcal{O}_Y \to 0.$$

For $\forall$ holo. line bundle L over X, we may think of it as a sheaf and tensor it w/ the above sequence, which yields another sequence

$$0 \to \mathcal{I}_Y \otimes L \to \mathcal{O}_X \otimes L \to \mathcal{O}_Y \otimes L \to 0.$$

$(\mathcal{I}_Y \otimes L)(U) = \{$ holo. section $s \in L(U)$ s.t. $s|_Y \equiv 0 \}$.

$\mathcal{O}_X \otimes L \cong L$

$\mathcal{O}_Y \otimes L =$ the restriction of L on Y, which is a holo. l.b. over Y.

$= i^*L$ where $i: Y \hookrightarrow X$ is the inclusion.

e.x. Check that the above sequence is exact as well.

Then one has

$$0 \to H^0(X, \mathcal{I}_Y \otimes L) \to H^0(X, L) \xrightarrow[\text{restriction}]{r} H^0(Y, i^*L) \to H^1(X, \mathcal{I}_Y \otimes L) \to \cdots$$

$Y \xrightarrow{i} X \xrightarrow{s} L$ Then r is surjective if $H^1(X, \mathcal{I}_Y \otimes L) = 0$.

$r(s) := i^*s$

▲ When Y is a cplx submanifold of codim 1, then

$\mathcal{I}_Y \underset{\text{as sheaves}}{\cong} \mathcal{O}(-Y)$. e.x. Check this. Hint: $\mathcal{O}(-Y)(U) = \{ f \in K(U), \operatorname{div}(f) \geq D \}$

In this case, r is surjective if $H^1(X, L \otimes \mathcal{O}(-Y)) = 0$.

The vanishing of this cohomology group is related to "vanishing thm's", which we will revisit in future courses.

- Normal bundle.

Let $Y \subseteq X$ be a cplx submfd of dim k. Then one has an exact sequence of hol. v.b.

$$0 \to TY^{1,0} \to TX^{1,0}|_Y \to N_Y \to 0,$$

where $N_Y = TX^{1,0}|_Y / TY^{1,0}$ is called the normal bundle of Y.

In local coordinates, one can explicitly calculate the transition matrix of N_Y as follows:

Choose $(U, z^1, \cdots, z^n)$ & $(V, w^1, \cdots, w^n)$ s.t.

$$\begin{cases} U_Y := U \cap Y = \{ z^{k+1} = \cdots = z^n = 0 \} \\ V_Y := V \cap Y = \{ w^{k+1} = \cdots = w^n = 0 \} \end{cases}$$

Let i, j denote indices in $\{1, \cdots, k\}$ & α, β in $\{k+1, \cdots, n\}$.

Then one has
$$\begin{cases} \dfrac{\partial}{\partial z^i} = \dfrac{\partial w^j}{\partial z^i} \dfrac{\partial}{\partial w^j} + \dfrac{\partial w^\alpha}{\partial z^i} \dfrac{\partial}{\partial w^\alpha} \\ \dfrac{\partial}{\partial z^\alpha} = \dfrac{\partial w^i}{\partial z^\alpha} \dfrac{\partial}{\partial w^i} + \dfrac{\partial w^\beta}{\partial z^\alpha} \dfrac{\partial}{\partial w^\beta} \end{cases}$$

Notice that $\dfrac{\partial w^\alpha}{\partial z^i} \equiv 0$ on $U_Y \cap V_Y$. $(w^\alpha(z^1, \cdots, z^k, 0, \cdots, 0) \equiv 0$ on $U_Y \cap V_Y)$

Thus the transition matrix of $TX^{1,0}|_Y$ is

$$g_{U_Y V_Y} = \begin{pmatrix} \dfrac{\partial w^j}{\partial z^i} & 0 \\ \dfrac{\partial w^i}{\partial z^\alpha} & \dfrac{\partial w^\beta}{\partial z^\alpha} \end{pmatrix}^{-1}$$

▲ If $\begin{pmatrix} A_1 & 0 \\ * & B_1 \end{pmatrix} \cdot \begin{pmatrix} A_2 & 0 \\ * & B_2 \end{pmatrix} \cdot \begin{pmatrix} A_3 & 0 \\ * & B_3 \end{pmatrix} = Id$ then $A_1 A_2 A_3 = B_1 B_2 B_3 = Id$.

Here $\left(\dfrac{\partial w^j}{\partial z^i}\right)^{-1}$ is the transition matrix for $TY^{1,0}$

& $\left(\dfrac{\partial w^\beta}{\partial z^\alpha}\right)^{-1}$ is the transition matrix for N_Y.

☆ Here we need to take the inverse as we are computing using "local frames" that give rise to local trivializations.

- As a consequence of the above discussion, one has

$$\det TX^{1,0}\big|_Y = \det TY^{1,0} \otimes \det N_Y.$$

In other words: $K_Y = K_X\big|_Y \otimes \det N_Y$.

This is called the adjunction formula.

- As a special case, when Y is of codim 1, one has

$$K_Y \cong (K_X + Y)\big|_Y.$$

Here $K_Y + Y$ is understood as $K_Y \otimes \mathcal{O}(Y)$, a line bundle on X.

▲ e.x. Check that $\mathcal{O}(Y)\big|_Y \cong N_Y$.

transition function of $\mathcal{O}(Y)$

Hint: $w^n(z^1, \cdots, z^{n-1}, 0) = 0 \Rightarrow \frac{\partial w^n}{\partial z^n}(z^1, \cdots, z^n, 0) = \frac{w^n}{z^n} = \left(\frac{z^n}{w^n}\right)^{-1}$

- Example. $X = \mathbb{CP}^n$. Y be a smooth hypersurface cut out by a homogeneous polynomial of degree d.

  Then $\mathcal{O}(Y) \cong \mathcal{O}(d)$. Recall that $K_X = \mathcal{O}(-n-1)$.

  Then $K_Y \cong \mathcal{O}(-n-1+d)\big|_Y$.

  - If $d = n+1$, then $K_Y \cong \mathcal{O}_Y$, so K_Y is trivial.

    In this case Y is a Calabi–Yau manifold.

  - If $d < n+1$, then $-K_Y \cong \mathcal{O}(n+1-d)\big|_Y$, so

    $-K_Y \cong \mathcal{O}_Y(D)$ where $D = H_{n+1-d} \cap Y$, H_{n+1-d} is a divisor cut out by a homogeneous polynomial of deg $n+1-d$.

    In this case Y is a Fano mfd.

  - If $d > n+1$, then $K_Y \cong \mathcal{O}(d-(n+1))\big|_Y$. Then

    $K_Y \cong \mathcal{O}_Y(D)$ w/ $D = H_{d-(n+1)} \cap Y$.

    In this case Y is of general type.

- Hermitian inner product.

Let $E \xrightarrow{\pi} X$ be a cplx vector bundle. A Hermitian metric h on E is a fiberwise Hermitian inner product on E_x for $\forall x \in X$ that varies smoothly in x.

Compare this w/ Riemannian metric.

Using local trivialization $\{U_i, \phi_i\}$, h is determined by

smooth map $h_i : U_i \to PH(r, \mathbb{C})$ ← positive definite Hermitian matrices

& $h_j = g_{ij}^* h_i g_{ij}$ on $U_i \cap U_j$, where $g_{ij} := \phi_i \circ \phi_j^{-1}$.

▲ So if $E = L$ is a cplx line bundle then one has
$h_j = |g_{ij}|^2 h_i$, where $h_i \in C^\infty(U_i, \mathbb{R}_{>0})$.

▲ As Riemannian metrics, Hermitian metrics always exist. !

▲ Assume that h' is another Hermitian metric on L, then locally, $h'_j = |g_{ij}|^2 h'_i$. So one has $\frac{h'_i}{h_i} = \frac{h'_j}{h_j}$.
thus $\frac{h'}{h}$ is a globally defined smooth positive function on X.
So we can write $h' = e^{-\phi} h$ for some $\phi \in C^\infty(X, \mathbb{R})$.
Thus, $\forall$ Hermitian metric on a cplx line bundle is of the form $e^{-\phi} h$, where h is some background metric.

Warning: This is not true for higher rank vector bundles.

- Assume that L is a holo. line bundle over a cplx mfd X. Let h be a Hermitian metric on L. Then the Chern curvature of h is defined to be

$$R_h := -\sqrt{-1}\,\partial\bar{\partial} \log h.$$

▲ Fact : R_h is well-defined, since locally one has

$$\sqrt{-1}\partial\bar\partial \log h_j = \sqrt{-1}\partial\bar\partial \log h_i + \sqrt{-1}\partial\bar\partial \log |g_{ij}|^2 = \sqrt{-1}\partial\bar\partial \log h_i .$$

So $\sqrt{-1}\partial\bar\partial \log h_i$ defines a global (1,1) form on X.

(e.x. Show that $\partial\bar\partial \log |f|^2 = 0$ for $\forall f \in \mathcal{O}^*(U)$)

▲ Recall that $\partial\bar\partial F := \frac{\partial^2 F}{\partial z^i \partial \bar z^j} dz^i \wedge d\bar z^j$.

▲ Since h is real valued, one easily check that

$$\overline{R_h} = R_h$$

So R_h is a real (1,1) form. Namely, in local real coord. R_h is a real 2-form.

▲ Also one has $dR_h = 0$. (This follows from $\partial^2 = \bar\partial^2 = 0$)

So R_h determines an element in the de Rham cohomology group i.e., $[R_h] \in H^2(X, \mathbb{R})$.

▲ $[R_h]$ is indpd of h, as a different h is given by

$h' = h e^{-\phi}$ So $R_{h'} = R_h + \sqrt{-1}\partial\bar\partial\phi = R_h + d\left(\frac{\partial - \bar\partial}{2\sqrt{-1}}(\phi)\right)$

Here $\frac{\partial-\bar\partial}{2\sqrt{-1}}\phi$ is a real 1-form. So $[R_{h'}] = [R_h]$.

▲ Let $\mathbb{Z} \xrightarrow{i} \mathbb{R}$ be the inclusion of constant sheaves.

This induces a map : $H^2(X, \mathbb{Z}) \xrightarrow{i_*} H^2(X, \mathbb{R})$, which will kill all the torsion part in $H^2(X, \mathbb{Z})$.

(Fact : $H^2(X, \mathbb{Z})$ = free part $\oplus$ torsion part)

▲ Fact One has $[R_h] = 2\pi\, i_* c_1(L)$. Thm 4.5 in [Wells book] Differential Analysis on cplx mfd.

One can compute $i_* c_1(L)$ using this identity.

In the literature, one often ignore i_* and identify

$c_1(L)$ w/ $i_* c_1(L)$

This is reasonable since $c_1(L^{\otimes k}) = i_* c_1(L^{\otimes k})$ for $\forall$ divisible $k \in \mathbb{N}_{>0}$

※ Pf of the above fact.

We fix a good cover $\{U_i\}$ s.t. $g_{ij} = e^{2\pi\sqrt{-1} f_{ij}}$ for some $f_{ij} \in \mathcal{O}(U_{ij})$. Then h is given by $\{h_i\}$ s.t. $h_j = |g_{ij}|^2 h_i$. Note that $|g_{ij}|^2 = e^{-4\pi \operatorname{Im} f_{ij}}$.

So one has $\log h_i - \log h_j = 4\pi \operatorname{Im} f_{ij}$.

Now, since f_{ij} is holomorphic, it is direct to check that

$$\frac{\sqrt{-1}}{2}(\partial - \bar\partial)(\log h_i - \log h_j) = 2\pi\sqrt{-1}(\partial - \bar\partial)\operatorname{Im} f_{ij} \overset{\text{Cauchy-Riemann}}{=} 2\pi\, d \operatorname{Re} f_{ij} \text{ on } U_{ij}$$

So from $\sqrt{-1}\partial\bar\partial \log h_i = d\left(\frac{\sqrt{-1}}{2}(\partial - \bar\partial)\log h_i\right)$, we see that

$$\theta_i := \frac{\sqrt{-1}}{2}(\partial - \bar\partial)(\log h_i - \log h_j) \text{ satisfies}$$

① $d\theta_i = R_h$ ② $\theta_i - \theta_j = 2\pi\, d \operatorname{Re} f_{ij}$.

Moreover on U_{ijk}, one has

$\operatorname{Re}(f_{ij} + f_{jk} + f_{ki}) = f_{ij} + f_{jk} + f_{ki} =: a_{ijk} \in \mathbb{Z}$ since $g_{ij} g_{jk} g_{ki} = 1$.

So the curvature 2-form R_h induces a cochain $\{2\pi a_{ijk}\}$.

This construction coincides w/ the map

$$H^2_{dR}(X, \mathbb{R}) \to \check{H}^2(X, \mathbb{R})$$

we discussed in Lecture 3.

$\square$.

- It is also clear from the above local construction that $\frac{1}{2\pi} R_h$ actually yields a cochain in $\check{H}^2(X, \mathbb{Z})$. So in particular $\frac{1}{2\pi} R_h$ is "integral", which implies that

$$\frac{1}{(2\pi)^n} \int_X R_{h_1} \wedge \cdots \wedge R_{h_n} \text{ is always an integer, for } \forall$$

n-line bundles $L_1, \cdots, L_n$ & $\forall$ Hermitian metrics $h_1, \cdots h_n$ on them.

- Now let us go back to the exact sequence

$$\cdots \to H^1(X,\mathcal{O}_X) \xrightarrow{\alpha} H^1(X,\mathcal{O}_X^*) \xrightarrow{c_1} H^2(X,\mathbb{Z}) \to \cdots$$

$\forall$ holo. line bundle L is locally given by $\{U_i \cap U_j, g_{ij}\}$ s.t. $g_{ij}\cdot g_{jk}\cdot g_{ki}=1$. One can choose sufficiently "fine" covering $\{U_i\}$ s.t. on $U_i \cap U_j$ one can write $g_{ij}=e^{2\pi\sqrt{-1}f_{ij}}$ for some $f_{ij}\in\mathcal{O}(U_i\cap U_j)$. Then cocycle condition for g_{ij} implies that $f_{ij}+f_{jk}+f_{ki}\in\mathbb{Z}$.
We put $a_{ijk}:=f_{ij}+f_{jk}+f_{ki}$. Then $\{U_i\cap U_j\cap U_k, a_{ijk}\}$ defines a cycle in $Z^2(\mathcal{U},\mathbb{Z})$. So it induces an element in $H^2(X,\mathbb{Z})$.
If L is induced from the map α, then a_{ijk} can be chosen to be 0.
So $c_1(L)=0$ in this case.

Since each f_{ij} is defined up to an additive integer a_{ijk} is defined up to an element in $B^2(\mathcal{U},\mathbb{Z})$.

* - In general c_1 is not surjective. But for $\forall\, \alpha\in H^2(X,\mathbb{Z})$ one can indeed construct a complex line bundle whose c_1 is α.

Using Čech cohomology, $\forall\alpha$ is represented by a cycle $\{U_{ijk}:=U_i\cap U_j\cap U_k,\ \alpha_{ijk}\in\mathbb{Z}\}$, s.t. $\alpha_{jkl}-\alpha_{ikl}+\alpha_{ijl}-\alpha_{ijk}=0$.
One define $f_{ij}\in C^\infty(U_i\cap U_j,\mathbb{R})$ by letting

$$f_{ij}:=\sum_k \alpha_{ijk}\theta_k$$ where $\{\theta_i\}$ is a partition of unity subodinate to $\{U_i\}$.

then one has

$$f_{ij}+f_{jk}+f_{ki}=\sum_l(\alpha_{ijl}+\alpha_{jkl}+\alpha_{kil})\theta_l$$

$$=\sum_l \alpha_{ijk}\theta_l=\alpha_{ijk}.$$

So letting $g_{ij}:=e^{2\pi\sqrt{-1}f_{ij}}$, one has $g_{ij}g_{jk}g_{ki}=1$.
This gives rise to a cplx line bundle on X.

• Example. $X = \mathbb{C}P^1$. $L = \mathcal{O}(-1)$. $z \in \mathbb{C} = X \setminus \{\infty\}$. $w \in \mathbb{C} = X \setminus \{0\}$. $zw = 1$.

Let $h = (1+|z|^2)$. Then h is a Hermitian metric on L.

$\mathcal{O}(-1) \subseteq \mathbb{P}^n \times \mathbb{C}^{n+1}$

We equip $\mathcal{O}(-1)$ w/ the standard Hermitian product on $\mathbb{C}^{n+1}$.

So one has

$$R_h = \sqrt{-1}\,\partial\bar\partial \log(1+|z|^2)$$
$$= -\sqrt{-1}\,\partial\left(\frac{z\,d\bar z}{1+|z|^2}\right)$$
$$= -\sqrt{-1}\,\frac{(1+|z|^2)dz\wedge d\bar z - |z|^2 dz\wedge d\bar z}{(1+|z|^2)^2}$$
$$= \frac{-\sqrt{-1}\,dz\wedge d\bar z}{(1+|z|^2)^2}.$$

If we use polar coord. $z = re^{i\theta}$, then

$$\sqrt{-1}\,dz\wedge d\bar z = \sqrt{-1}\left(e^{i\theta}dr + \sqrt{-1}\,re^{i\theta}d\theta\right)\wedge\left(e^{-i\theta}dr - \sqrt{-1}\,re^{-i\theta}d\theta\right)$$
$$= 2r\,dr\,d\theta.$$

So $R_h = -\dfrac{2r\,dr\,d\theta}{(1+r^2)^2}$. This is indeed a real 2-form. It is negatively definite! $-R_h$ is a Riem. metric

▲ Compute $\frac{1}{2\pi}\int_X R_h = \frac{1}{2\pi}\int_{\mathbb{C}} -\frac{2r\,dr\,d\theta}{(1+r^2)^2} = -1$.

Thus $[X]\cdot c_1(L) = -1$. (There is no torsion in $H^2(X,\mathbb{Z})$)

This is why L is denoted by $\mathcal{O}(-1)$, as it has degree -1.

▲ $h^{-1} := \frac{1}{1+|z|^2}$ is a metric on $\mathcal{O}(1)$.

$R_{h^{-1}} = -R_h$. So $\frac{1}{2\pi}\int_X R_{h^{-1}} = 1$. This explains why the dual of $\mathcal{O}(-1)$ is denoted by $\mathcal{O}(1)$.

So $c_1(L)$ is a generator of $H^2(X,\mathbb{Z}) \simeq \mathbb{Z}$.

▲ h^k is a metric on $\mathcal{O}(-k)$ for $\forall k \in \mathbb{Z}$.

& $\frac{1}{2\pi}\int_X R_{h^k} = \frac{1}{2\pi}\int_X k R_h = -k$.

※ 遗计

① Consider the anti-canonical line bundle $-K_X$. let h be any Hermition metric on $-K_X$. Then

$$\frac{1}{2\pi}[R_h] = c_1(X).$$ (up to some torsion).

Note that $\forall$ Hermitian metric on $-K_X$ can be identified w/ a smooth positive volume form on X.

Indeed, locally on $(U, z^1, \cdots, z^n)$ & $(V, w^1, \cdots, w^n)$ one has

$$h_V = \left|\det\left(\frac{\partial z^i}{\partial w_j}\right)\right|^2 h_U.$$ This implies that

$$h_U \, dz^1\wedge\cdots\wedge dz^n \wedge d\bar{z}^1\wedge\cdots\wedge d\bar{z}^n = h_V \, dw^1\wedge\cdots\wedge dw^n\wedge d\bar{w}^1\wedge\cdots\wedge d\bar{w}^n$$

So h defines a global smooth positive (n,n) form, which serves as a volume form.

Conversely $\forall$ smooth positive volume form Ω gives a Hermitian metric on $-K_X$.

So one can compute $c_1(X)$ using volume forms.

Then the Chern curvature R_Ω is called the Ricci form of Ω

② Intersection number. Given n holo. l.b. $L_1, \cdots, L_n$

Define $$L_1 \cdot L_2 \cdot \cdots \cdot L_n := \frac{1}{(2\pi)^n}\int_X R_{h_1}\wedge R_{h_2}\wedge\cdots\wedge R_{h_n}$$

Here $h_1, \cdots, h_n$ are arbitrary Hermitian metrics on $L_1, \cdots, L_n$, resp.

This definition is indpd of the choice of h_i's (By Stoke's thm)

▲ One always has $L_1 \cdots L_n \in \mathbb{Z}$ since $c_1(L) \in H^2(X, \mathbb{Z})$.