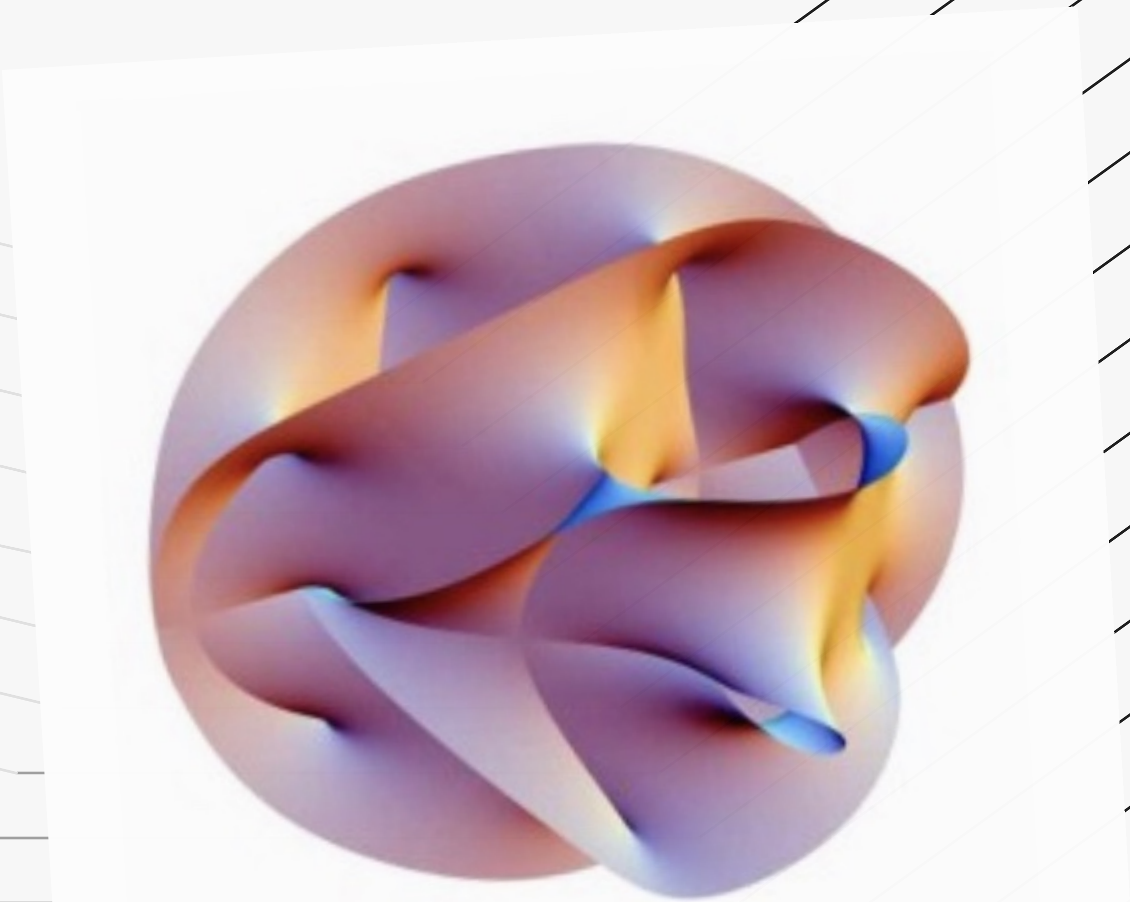
Lecture 7

Kähler Manifolds



Outline

1. Definition & examples
2. Basic properties $\begin{cases} H^{2k} \neq 0 \\ \text{volume form.} \end{cases}$
3. Levi-Civita connection
4. Curvature $\begin{cases} HK \\ BK \\ Ric \\ R \end{cases}$

- Setup: Let X be a cplx mfd, w/ complex structure J of dim n.
  Here recall that $J^2 = -id$ & J induces ∂ & $\bar\partial$ operators on X s.t. $d = \partial + \bar\partial$. And $TX^{\mathbb{C}} = TX^{1,0} \oplus TX^{0,1}$

- Note that X itself is also a diff. mfd so we may consider Riemannian metrics on X. A Riem. metric g on X is called "Hermitian" if for $\forall\, x \in X$, $\forall\, u, v \in T_xX$, one has
$$g(Ju, Jv) = g(u, v),$$
namely g is J-invariant.

- Hermitian metrics always exists. This is because locally the standard Euclidean metric g_E on $\mathbb{C}^n$ is Hermitian w.r.t. the standard cplx stru. So we can use partition of unity to construct global Hermitian metrics on X. In what follows we will always assume that g is Hermitian.

- Why the name "Hermitian"? This is because g induces a Hermitian metric h on the cplx v.b. $TX^{\mathbb{C}}$ in the following way.
  - Step 1. Extend g $\mathbb{C}$-linearly to $TX^{\mathbb{C}}$

    Then we find that $\begin{cases} g(u, v) = 0 & \text{for } \forall\, u, v \in TX^{1,0}. \\ g(u, v) = 0 & \text{for } \forall\, u, v \in TX^{0,1}. \end{cases}$

  - Step 2. Define h by putting
$$h(u, v) := g(u, \bar v) \quad \text{for } \forall\, u, v \in TX^{\mathbb{C}}.$$
    Then h is positive definite. & $h(u, v) = \overline{h(v, u)}$.

    $\forall\, u \in T_X^{1,0}$ can be written as $u = \mathrm{Re}\, u + \sqrt{-1}\, \mathrm{Im}\, u$.
    then $g(u, \bar u) = g(\mathrm{Re}\, u + \sqrt{-1}\, \mathrm{Im}\, u, \mathrm{Re}\, u - \sqrt{-1}\, \mathrm{Im}\, u)$
$$= \|\mathrm{Re}\, u\|^2 + \|\mathrm{Im}\, u\|^2.$$

Using h, one sees that $h(u,v)=0$ for $\forall u\in TX^{1,0}$, $\forall v\in TX^{0,1}$
So we see that h induces an orthogonal decomposition:
$$TX^{\mathbb{C}} = TX^{1,0}\oplus TX^{0,1} \quad \text{w/} \quad TX^{1,0}\perp_h TX^{0,1}.$$

- The Kähler form (fundamental form) of (X, g, J) is defined by $\omega := g(J\cdot,\cdot)$. We will see below that ω is a positive real $(1,1)$ form.

- Now let us compute everything locally. Choose a holo coord. $(z^1,\dots,z^n)$ write $z^i = x^i+\sqrt{-1}\,y^i$. Assume that g is given by
$$g = g_{ij}\,dx^i\otimes dx^j + g_{i\bar j}\,dx^i\otimes dy^j + g_{\bar i j}\,dy^i\otimes dx^j + g_{\bar i\bar j}\,dy^i\otimes dy^j$$
Then both (g_{ij}) & $(g_{\bar i\bar j})$ are symmetry & positive definite & $g_{i\bar j} = g_{\bar j i}$

Recall that $J\,dx^i = -dy^i$ & $J\,dy^i = dx^i$.

So $g(J\cdot,J\cdot)=g(\cdot,\cdot)$ implies that

$$g = g_{ij}\,dy^i\otimes dy^j - g_{i\bar j}\,dy^i\otimes dx^j - g_{\bar i j}\,dx^i\otimes dy^j + g_{\bar i\bar j}\,dx^i\otimes dx^j$$

thus $\begin{cases} g_{ij} = g_{\bar i\bar j} \\ g_{i\bar j} = -g_{\bar i j} = -g_{\bar j i}\end{cases} \Rightarrow (g_{i\bar j})$ is skew-symmetry. $\boxed{g=\begin{pmatrix} A & B \\ -B & A\end{pmatrix}}$

So in conclusion $(g_{ij}) = (g_{\bar i\bar j})>0$ & $(g_{i\bar j}) = -(g_{\bar i j}) = -(g_{i\bar j})^T$.
Now we write out everything using dz^i & $d\bar z^j$.
Plugging in $dx^i = \frac12(dz^i+d\bar z^i)$ & $dy^i = -\frac{\sqrt{-1}}{2}(dz^i - d\bar z^i)$ in g we find
$$g = \frac14\Big[g_{ij}(dz^i+d\bar z^i)\otimes(dz^j+d\bar z^j) - \sqrt{-1}\,g_{i\bar j}(dz^i+d\bar z^i)\otimes(dz^j-d\bar z^j)$$
$$- \sqrt{-1}\,g_{\bar i j}(dz^i-d\bar z^i)\otimes(dz^j+d\bar z^j) - g_{\bar i\bar j}(dz^i-d\bar z^i)\otimes(dz^j-d\bar z^j)\Big].$$
$$= \frac14\Big[(g_{ij}-g_{\bar i\bar j}-\sqrt{-1}\,g_{i\bar j}-\sqrt{-1}\,g_{\bar i j})\,dz^i\otimes dz^j + (g_{ij}+\sqrt{-1}\,g_{i\bar j}-\sqrt{-1}\,g_{\bar i j}+g_{\bar i\bar j})\,dz^i\otimes d\bar z^j$$
$$+ (g_{ij}-\sqrt{-1}\,g_{i\bar j}+\sqrt{-1}\,g_{\bar i j}+g_{\bar i\bar j})\,d\bar z^i\otimes dz^j + (g_{ij}-\sqrt{-1}\,g_{i\bar j}+\sqrt{-1}\,g_{\bar i j}-g_{\bar i\bar j})\,d\bar z^i\otimes d\bar z^j\Big].$$
$$= \frac12(g_{ij}+\sqrt{-1}\,g_{i\bar j})\,dz^i\otimes d\bar z^j + \frac12(g_{ij}-\sqrt{-1}\,g_{i\bar j})\,d\bar z^i\otimes dz^j.$$

$$\Rightarrow h\Big(\frac{\partial}{\partial z^i},\frac{\partial}{\partial \bar z^j}\Big) = \frac12(g_{ij}+\sqrt{-1}\,g_{i\bar j}) \ \&\ h\Big(\frac{\partial}{\partial \bar z^i},\frac{\partial}{\partial z^j}\Big) = \frac12(g_{ij}-\sqrt{-1}\,g_{i\bar j})$$

So as a Hermitian inner product on $TX^{\mathbb{C}}$, h is given by

$$h = \frac{1}{2}\begin{pmatrix} H & 0 \\ 0 & \overline{H} \end{pmatrix}$$ w.r.t. the basis $\left\{\frac{\partial}{\partial z^1},\dots,\frac{\partial}{\partial z^n},\frac{\partial}{\partial \bar z^1},\dots,\frac{\partial}{\partial \bar z^n}\right\}$.

We further compute ω. Using $J\,dz^i = \sqrt{-1}\,dz^i$ & $J\,d\bar z^j = -\sqrt{-1}\,d\bar z^j$.

$$\omega = g(J\cdot,\cdot) = \frac{\sqrt{-1}}{2}(g_{ij}+\sqrt{-1}g_{i\bar j})\,dz^i\otimes d\bar z^j - \frac{\sqrt{-1}}{2}(g_{ij}-\sqrt{-1}g_{i\bar j})\,d\bar z^i\otimes dz^j$$

This computation shows that ω is real, positive of type $(1,1)$.

$$\overset{\text{swap } i,j}{=} \frac{\sqrt{-1}}{2}(g_{ij}+\sqrt{-1}g_{i\bar j})\,dz^i\otimes d\bar z^j - \frac{\sqrt{-1}}{2}(\underbrace{g_{ji}}_{=g_{ij}}-\sqrt{-1}\underbrace{g_{j\bar i}}_{=-g_{i\bar j}})\,d\bar z^j\otimes dz^i$$

$$= \frac{\sqrt{-1}}{2}(g_{ij}+\sqrt{-1}g_{i\bar j})\left(dz^i\otimes d\bar z^j - d\bar z^j\otimes dz^i\right)$$

$$= \frac{\sqrt{-1}}{2}(g_{ij}+\sqrt{-1}g_{i\bar j})\,dz^i\wedge d\bar z^j. \quad //$$

Put $h_{i\bar j} := g_{ij}+\sqrt{-1}g_{i\bar j}$. Then $H := (h_{i\bar j})$ is positive Hermitian. $H^T = \overline{H}$

▲ Note that one can diagonalize $h_{i\bar j}$ at one pt $\in X$ after a linear transformation of $z^1,\dots,z^n$. In this case one has

$g_{ij} = \lambda_i$ & $g_{i\bar j} = 0$. So g is also diagonalized.

$$g = \begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & 0 & \\ & & \lambda_n & & & \\ & & & \lambda_1 & & \\ & 0 & & & \ddots & \\ & & & & & \lambda_n \end{pmatrix} \qquad H = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}.$$

Thus we find that $\det H = \sqrt{\det g}$. This holds everywhere.

One can also see this w/o diagonalization, as $g = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$ $H = A+\sqrt{-1}B$.

So $\det g = |\det(A+\sqrt{-1}B)|^2 = (\det H)^2$.

Exam Show that $\boxed{\sqrt{\det g}\,dx^1\wedge dy^1\wedge\cdots\wedge dx^n\wedge dy^n = \frac{\omega^n}{n!}}$. $\quad \omega^n := \underbrace{\omega\wedge\cdots\wedge\omega}_{n}$

So in particular, ω^n defines a volume form on X.

⌈ After a coord. change, we have $g = \sum_i dx^i\otimes dx^i + \sum_i dy^i\otimes dy^i$.

& $\omega = \frac{\sqrt{-1}}{2}\sum_i dz^i\wedge d\bar z^i \Rightarrow \omega^n = n!\left(\frac{\sqrt{-1}}{2}dz^1\wedge d\bar z^1\right)\wedge\cdots\wedge\left(\frac{\sqrt{-1}}{2}dz^n\wedge d\bar z^n\right)$

$= n!\,dx^1\wedge dy^1\wedge\cdots\wedge dx^n\wedge dy^n$. ⌋

▲ Since ω^n is a volume form, it is also a Hermitian metric for $-K_X$.

Then the Chern curvature $-\sqrt{-1}\partial\bar\partial\log\det\omega \in 2\pi c_1(X)$.

So one can compute $c_1(X)$ using Kähler form ω induced from (X,g,J).

★ The above discussion holds for all cplx mfd.

$\frac{\partial h_{i\bar{j}}}{\partial \bar{z}^k} = \frac{\partial h_{i\bar{k}}}{\partial \bar{z}^j}$ & $\frac{\partial h_{i\bar{j}}}{\partial z^k} = \frac{\partial h_{k\bar{j}}}{\partial z^i}$.

- **Def**. (X, g, J) is called Kähler if $d\omega = 0$, where $\omega = g(J\cdot, \cdot)$.
  ① For simplicity, a Kähler mfd will often be denoted by (X, ω) when the underlying cplx struc. J is fixed; in this setting $g = \omega(\cdot, J\cdot)$ & g is called a Kähler metric.
  ② A cplx mfd is called Kähler if $\exists$ Kähler metric on X.

*. Kähler condition is quite special. Although a cplx mfd is always "locally Kähler", one cannot path these local Kähler metrics using partition of unity, as the d-closedness is not likely to be preserved.

- Examples

① $\mathbb{C}^n$ equipped w/ the Euclidean metric g_E is Kähler.
in this case $\omega = \frac{\sqrt{-1}}{2}(dz^1 \wedge d\bar{z}^1 + \cdots + dz^n \wedge d\bar{z}^n)$,
So clearly $d\omega = 0$.

② $B_1 \subseteq \mathbb{C}^n$ is Kähler since $\omega := \sqrt{-1}\partial\bar{\partial}\log\frac{1}{1-|z|^2} > 0$. complex hyperbolic metric.

③ $\forall$ cplx submfd of a Kähler mfd is Kähler.

④ Assume that L is a holo. line bundle on a cplx mfd. Assume that L admits a Hermitian metric h s.t. its Chern curvature form $R_h := -i\partial\bar{\partial}\log h$ is positive definite, meaning that, writting $R_h = \sqrt{-1}a_{i\bar{j}}\, dz^i \wedge d\bar{z}^j$ locally, $(a_{i\bar{j}})$ is a positive definite Hermitian matrix.
In this case, R_h defines a Kähler metric $g := R_h(\cdot, J\cdot)$.
So X is Kähler. (We will see that X is actually projective)

⑤ $\mathbb{CP}^n$ is Kähler.
To see this, we consider $L := \mathcal{O}(1)$ on X. Then we show that $h := \frac{1}{(|z_0|^2 + \cdots + |z_n|^2)}$, as a Hermitian metric on L gives rise to a Kähler metric on $\mathbb{CP}^n$.
On $U_0 := \{z_0 \neq 0\}$, $R_h = \sqrt{-1}\partial\bar{\partial}\log(1 + |\xi_1|^2 + \cdots + |\xi_n|^2)$, here $\xi_i = \frac{z_i}{z_0}$. We claim that $R_h > 0$.

To see this, we use the fact that $\mathbb{CP}^n$ is homogeneous so we can assume that we are computing at $[1:0:\cdots:0]$.
At this pt $R_h = \sqrt{-1}(dz_1 \wedge d\bar{z}_1 + \cdots + dz_n \wedge d\bar{z}_n) > 0$.
The induced Kähler metric has a special name, which is called Fubini-Study metric & R_h is denoted as ω_{FS}.

⑥ $\forall$ cplx submfd of $\mathbb{CP}^n$ is Kähler.
So in particular $\forall$ projective mfd is Kähler.

⑦ S^{2n} is NOT Kähler where $n \geq 2$.
Assume otherwise S^{2n} admits a d-closed Kähler form ω.
Then it induces an element $[\omega] \in H^2_{dR}(S^{2n}, \mathbb{R})$
Since $H^2_{dR}(S^{2n}, \mathbb{R}) \cong H^2(S^{2n}, \mathbb{R}) = 0$ for $n \geq 2$, we see that $\omega = d\theta$ for some 1-form θ on S^{2n}.
Thus $\omega^n = d\theta \wedge \cdots \wedge d\theta = d(\theta \wedge d\theta \wedge \cdots \wedge d\theta)$.
So $\int_{S^{2n}} \omega^n = \int_{S^{2n}} d(\theta \wedge d\theta \wedge \cdots \wedge d\theta) = 0$ by Stokes' thm, which is impossible since ω^n is a volume form.

▲ For the same reason, Hopf surface $X \cong S^1 \times S^3$ is not Kähler.

▲ Exam. If X is a cpt Kähler mfd, then $H^{2i}(X, \mathbb{R}) \neq 0$ for $\forall\, i = 0, 1, \cdots, n$.
Pf: $H^0(X, \mathbb{R}) \neq 0$ clearly. For $i \geq 1$, consider ω^i, then $d\omega^i = 0$.
If $\omega^i = d\theta$ for some $(2i-1)$-form, then $\omega^n = d(\theta \wedge \omega^{n-i})$,
So $\int_X \omega^n = 0$, which is absurd. $\square$

▲ We will see, using Hodge theory, that $\dim H^{2i+1}(X, \mathbb{C})$ is always even.

So being Kähler is a very restrictive condition.
In some sense, Kähler is very closed to being "algebraic".

- In what follows we fix a Kähler mfd (X, g, J), whose associated Kähler form is denoted by ω. Let ∇ denote the Levi-Civita connection of g. One can extend ∇ by $\mathbb{C}$-linearity so that it defines a connection on $TX^{\mathbb{C}}$.

We put for simplicity $\partial_i := \frac{\partial}{\partial z^i}$ & $\partial_{\bar j} := \frac{\partial}{\partial \bar z^j}$. Then ∇ is determined by

$$\begin{cases} \nabla_{\partial_i}\partial_j = \Gamma_{ij}^{k}\partial_k + \Gamma_{ij}^{\bar k}\partial_{\bar k} \\ \nabla_{\partial_{\bar i}}\partial_j = \Gamma_{\bar i j}^{k}\partial_k + \Gamma_{\bar i j}^{\bar k}\partial_{\bar k} \\ \nabla_{\partial_i}\partial_{\bar j} = \Gamma_{i\bar j}^{k}\partial_k + \Gamma_{i\bar j}^{\bar k}\partial_{\bar k} \\ \nabla_{\partial_{\bar i}}\partial_{\bar j} = \Gamma_{\bar i\bar j}^{k}\partial_k + \Gamma_{\bar i\bar j}^{\bar k}\partial_{\bar k}. \end{cases}$$

Since ∇ is real, $\overline{\Gamma_{ij}^{k}} = \Gamma_{\bar i\bar j}^{\bar k}$, $\overline{\Gamma_{ij}^{\bar k}} = \Gamma_{\bar i\bar j}^{k}$, $\overline{\Gamma_{i\bar j}^{k}} = \Gamma_{\bar i j}^{\bar k}$ & $\overline{\Gamma_{i\bar j}^{\bar k}} = \Gamma_{\bar i j}^{k}$.

So it is enough to consider Γ_{ij}^{k}, $\Gamma_{ij}^{\bar k}$, $\Gamma_{\bar i j}^{k}$ & $\Gamma_{\bar i j}^{\bar k}$.

Since ∇ is torsion free, we have $\begin{cases} \Gamma_{ij}^{k} = \Gamma_{ji}^{k}, \quad \Gamma_{i\bar j}^{k} = \Gamma_{\bar j i}^{k}. \\ \Gamma_{\bar j i}^{k} = \overline{\Gamma_{\bar i j}^{\bar k}}, \quad \Gamma_{\bar i j}^{k} = \overline{\Gamma_{\bar j i}^{\bar k}} \end{cases}$

We write $\omega = \sqrt{-1}\,\omega_{i\bar j}\, dz^i \wedge d\bar z^j$.

So $g = \omega_{i\bar j}\, dz^i \otimes d\bar z^j + \omega_{j\bar i}\, d\bar z^i \otimes dz^j$

Since $\nabla g = 0$, we have

$$0 = \partial_i \underbrace{g(\partial_k, \partial_l)}_{=0} = g(\nabla_{\partial_i}\partial_k, \partial_l) + g(\partial_k, \nabla_{\partial_i}\partial_l)$$

$$= \Gamma_{ik}^{\bar q}\omega_{l\bar q} + \Gamma_{il}^{\bar q}\omega_{k\bar q}.$$

Exchanging i & k we have $0 = \Gamma_{ik}^{\bar q}\omega_{l\bar q} + \Gamma_{kl}^{\bar q}\omega_{i\bar q}$ as well.

$$\Rightarrow \Gamma_{il}^{\bar q}\omega_{k\bar q} = \Gamma_{kl}^{\bar q}\omega_{i\bar q}. \text{ Thus}$$

$$\Gamma_{il}^{\bar q}\omega_{k\bar q} = \Gamma_{li}^{\bar q}\omega_{k\bar q} = \Gamma_{ki}^{\bar q}\omega_{l\bar q} = \Gamma_{ik}^{\bar q}\omega_{l\bar q}.$$

$$\Rightarrow \Gamma_{ik}^{\bar q}\omega_{l\bar q} = 0 \Rightarrow \Gamma_{ik}^{\bar q} = 0. \text{ So } \nabla_{\partial_i}\partial_j = \Gamma_{ij}^{k}\partial_k.$$

On the other hand

$$\frac{\partial \omega_{k\bar l}}{\partial z^i} = \partial_i g(\partial_k, \partial_{\bar l}) = g(\nabla_{\partial_i}\partial_k, \partial_{\bar l}) + g(\partial_k, \underset{\nabla_{\partial_{\bar l}}\partial_i}{\underset{\parallel}{\nabla_{\partial_i}\partial_{\bar l}}})$$

$$= \Gamma_{ik}^{p}\omega_{p\bar l} + \Gamma_{\bar l i}^{\bar q}\omega_{k\bar q}$$

Exchanging k & i we deduce that $\Gamma_{\bar l i}^{\bar q}\omega_{k\bar q} = \Gamma_{\bar l k}^{\bar q}\omega_{i\bar q}$.

Using $\partial_{\bar l}\, g(\partial_i, \partial_k) = 0$ we find that $\Gamma_{\bar l i}^{\bar q}\omega_{k\bar q} = 0$

Thus $\Gamma_{\bar{\ell}i}^{\bar{q}} = 0 \Rightarrow \Gamma_{\bar{\ell}i}^{q} = \overline{\Gamma_{\bar{i}\ell}^{\bar{q}}} = 0 \Rightarrow \nabla_{\partial_i}\partial_{\bar{j}} = 0$.

So finaly, we arrive at $\frac{\partial \omega_{k\bar{\ell}}}{\partial z^i} = \Gamma_{ik}^{p}\,\omega_{p\bar{\ell}}$

$\Rightarrow \Gamma_{ik}^{p} = \omega^{p\bar{q}} \frac{\partial \omega_{i\bar{q}}}{\partial z^k}$. So the Levi-Civita connection of g is explicitly given by the condition

$$\begin{cases} \nabla_{\partial_i}\partial_j = \omega^{p\bar{q}} \dfrac{\partial \omega_{i\bar{q}}}{\partial z^j}\,\partial_p. \\ \nabla_{\partial_i}\partial_{\bar{j}} = 0. \end{cases}$$

So using the Kähler form $\omega = \sqrt{-1}\,\omega_{i\bar{j}}\, dz^i \wedge d\bar{z}^j$ one can easily compute the Christoffel symbol. This, again, shows that the geometry of a Kähler mfd is completely determined by its Kähler form.

▲ By abuse of language, we will also call ω "Kähler metric." And locally we will write

$$\omega = \sqrt{-1}\, g_{i\bar{j}}\, dz^i \wedge d\bar{z}^j.$$

Then the corresponding Riem metric g is given by

$$g = g_{i\bar{j}}\, dz^i \otimes d\bar{z}^j + g_{j\bar{i}}\, d\bar{z}^i \otimes dz^j.$$

- <u>Lem</u>. Let (X,ω) be a Kähler mfd. Then for $\forall p \in X$, $\exists$ local holo. coord. $(z^1,\cdots,z^n)$ around p s.t.

$$\omega = \sqrt{-1}\,(\delta_{ij} + O(|z|^2))\, dz^i \wedge d\bar{z}^j.$$

pf: First, one can choose $(w^1,\cdots,w^n)$ around p s.t.

$$\omega = \sqrt{-1}\, g_{i\bar{j}}\, dw^i \wedge d\bar{w}^j \quad w/ \quad g_{i\bar{j}} = \delta_{ij} + \frac{\partial g_{i\bar{j}}}{\partial w^k}(p)\, w^k + \frac{\partial g_{i\bar{j}}}{\partial \bar{w}^l}(p)\, \bar{w}^l + O(|w|^2)$$

Put $a_{ikj} := -\frac{\partial g_{i\bar{j}}}{\partial w^k}(p)$. Then $a_{ikj} = a_{kij}$.

We set $z^i := w^i - \frac{1}{2} a_{kji} w_k w_j$. Then $(z^1,\cdots,z^n)$ is a holo. coord. s.t. $w^i = z^i + \frac{1}{2} a_{kji}\, z^k z^j + O(|z|^2)$

So that $dw^i = dz^i + a_{kji}\, z^k dz^j$ & $d\bar{w}^j = d\bar{z}^j + \bar{a}_{stj}\, \bar{z}^s d\bar{z}^t$.

Plugging these into the expression of ω,

$$\omega=\sqrt{-1}\left(\delta_{ij}+\frac{\partial g_{i\bar{j}}}{\partial w^k}(p)z^k+\frac{\partial g_{i\bar{j}}}{\partial \bar{w}^l}(p)\bar{z}^l+O(|z|^2)\right)\left(dz^i+a_{kpi}z^k dz^p\right)\wedge\left(d\bar{z}^j+\overline{a_{stj}}\bar{z}^s d\bar{z}^t\right)$$

$$=\sqrt{-1}\left[\delta_{ij}+\left(\frac{\partial g_{i\bar{j}}}{\partial w^k}(p)+a_{kij}\right)z^k+\left(\frac{\partial g_{i\bar{j}}}{\partial \bar{w}^l}(p)+\overline{a_{lji}}\right)\bar{z}^l+O(|z|^2)\right)dz^i\wedge d\bar{z}^j.$$

$$=\sqrt{-1}\left(\delta_{ij}+O(|z|^2)\right)dz^i\wedge d\bar{z}^j.$$

$\square$

- <u>Cor</u>. Let (X,ω) be a Kähler mfd, then for $\forall\, p\in X$ $\exists$ holo. coord. $(z^1,\cdots,z^n)$ around p s.t.

  $\nabla_{\partial_i}\partial_j=0$ at p.

  Such coord. is called "Kähler normal coord. system".

  ▲ In general this is not true for cplx mfd! So this is another special propert of Kähler mfd.

- <u>Cor</u>. If (X,g,J) is Kähler, then $\nabla J=0$.

  pf: In Kähler normal coord. J has const. coefficients:

  $J=\sqrt{-1}\,dz^i\otimes\frac{\partial}{\partial z^i}-\sqrt{-1}\,d\bar{z}^i\otimes\frac{\partial}{\partial \bar{z}^i}$ (this holds in $\forall$ holo. coord. system)

  & $\nabla J(p)=0$. Thus $\nabla J=0$ everywhere. $\square$

  ▲ $\nabla J=0$ implies that $\nabla(JV)=J\nabla V$ for $\forall$ real v.f. V.

- <u>Cor</u>. If (X,g,J) is Kähler, then $\begin{cases} R(U,V)JW=JR(U,V)W. \\ R(JU,JV)W=R(U,V)W. \end{cases}$

pf: ① $\big(\underbrace{\nabla_U\nabla_V-\nabla_V\nabla_U-\nabla_{[U,V]}}_{=R(U,V)}\big)JW=J\left(\nabla_U\nabla_V-\nabla_V\nabla_U-\nabla_{[U,V]}\right)W$

② For $\forall Z$, $g\left(R(JU,JV)W,Z\right)=R(JU,JV,Z,W)$

$=R(Z,W,JU,JV)=R(Z,W,U,V)$ (g is Hermitian)

$=R(U,V,Z,W)=g(R(U,V)W,Z)$. $\square$

- Extending R $\mathbb{C}$-linearily we get a 4-tensor on $TX^{\mathbb{C}}$.
  For $\forall\ u,v \in TX^{1,0}$ one has $R(U,V) = R(JU,JV) = -R(U,V)$
  So $R(U,V)=0$. Similarily $R(U,V)=0$ for $\forall\ u,v \in TX^{0,1}$.
  Thus the only interesting component of R is

$$R_{i\bar{j}k\bar{l}} := g\left(R(\partial_i, \partial_{\bar{j}})\partial_k, \partial_{\bar{l}} \right)$$

This convention is different from the Riem. case!

$$= g\left((\nabla_i \nabla_{\bar{j}} - \nabla_{\bar{j}} \nabla_i)\partial_k, \partial_{\bar{l}} \right) = -g\left(\nabla_{\bar{j}}(\Gamma_{ik}^s \partial_s), \partial_{\bar{l}} \right)$$

Assume $\omega = \sqrt{-1} g_{i\bar{j}} dz^i \wedge d\bar{z}^j$.

$$= -\frac{\partial \Gamma_{ik}^s}{\partial \bar{z}^j} g_{s\bar{l}} = -\frac{\partial \left(g^{s\bar{t}} \frac{\partial g_{i\bar{t}}}{\partial z^k} \right)}{\partial \bar{z}^j} g_{s\bar{l}}$$

$$= -\frac{\partial^2 g_{i\bar{j}}}{\partial z^k \partial \bar{z}^l} + g^{p\bar{q}} \frac{\partial g_{i\bar{q}}}{\partial z^k} \frac{\partial g_{p\bar{j}}}{\partial \bar{z}^l}.$$

▲ $R_{i\bar{j}k\bar{l}}$ satisfies $R_{i\bar{j}k\bar{l}} = R_{k\bar{j}i\bar{l}} = R_{i\bar{l}k\bar{j}} = R_{k\bar{l}i\bar{j}}$

- Define $R_{i\bar{j}} := g^{k\bar{l}} R_{i\bar{j}k\bar{l}}$. Then $\mathrm{Ric}(\omega) := \sqrt{-1} R_{i\bar{j}} dz^i \wedge d\bar{z}^j$
  is called the Ricci form of ω.
  One actually has

$$R_{i\bar{j}} = -g^{k\bar{l}} \frac{\partial^2 g_{k\bar{l}}}{\partial z^i \partial \bar{z}^j} + g^{k\bar{l}} g^{p\bar{q}} \frac{\partial g_{k\bar{q}}}{\partial z^i} \frac{\partial g_{p\bar{l}}}{\partial \bar{z}^j}$$

$$= \frac{\partial}{\partial z^i}\left(-g^{k\bar{l}} \frac{\partial g_{k\bar{l}}}{\partial \bar{z}^j} \right)$$

$$= -\frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log \det (g_{k\bar{l}})$$

So we find that $\mathrm{Ric}(\omega) = -\sqrt{-1}\partial\bar{\partial} \log \det \omega$. d-closed real (1,1) form

This in particular shows that $\mathrm{Ric}(\omega)$ is the Chern curvature of the Hermitian metric $\det \omega$ on K_X.

So we have $[\mathrm{Ric}(\omega)] = 2\pi C_1(X)$.

• **Thm.** Let $Ric(\cdot,\cdot)$ denote the Ricci tensor of the Kähler metric g. Then ① $Ric(J\cdot, J\cdot) = Ric(\cdot,\cdot)$

② $Ric(\omega) = Ric(J\cdot, \cdot)$.

pf: ① For $\forall X, Y \in \Gamma(X, TX)$, one has $\{e_i\}_{i=1}^{2n}$ local orthonormal frame

$$Ric(JX, JY) := g(R(JX, e_i)e_i, JY)$$
$$= -g(R(JX, e_i)Je_i, Y) = g(R(X, Je_i)Je_i, Y)$$
$$= Ric(X, Y).$$

② We compute using Kähler normal coord $(z^1, \cdots, z^n)$ around p.

Also write $z^i = x^i + \sqrt{-1}y^i$. Then one has (Using the local computation for g & ω)

$g = 2\sum_i dx^i \otimes dx^i + 2\sum_i dy^i \otimes dy^i$ at p.

Write $e_i = \frac{1}{\sqrt{2}}\frac{\partial}{\partial x^i}$ & $Je_i = \frac{1}{\sqrt{2}}\frac{\partial}{\partial y^i}$.

then $\{e_1, \cdots, e_n, Je_1, \cdots, Je_n\}$ is orthonormal at p.

Since $Ric(J\cdot, J\cdot) = Ric(\cdot,\cdot)$, one can write

$Ric(J\cdot, \cdot) = \sqrt{-1}\,\theta_{i\bar{j}}\, dz^i \wedge d\bar{z}^j$, then (again using the local computation for g)

$$\theta_{i\bar{j}} = Ric(\partial_i, \partial_{\bar{j}}) = \sum_\alpha g(R(\partial_i, e_\alpha)e_\alpha, \partial_{\bar{j}})$$
$$+ \sum_\alpha g(R(\partial_i, Je_\alpha)Je_\alpha, \partial_{\bar{j}})$$
$$= \sum_\alpha \left(g(R(\partial_i, e_\alpha)e_\alpha, \partial_{\bar{j}}) + \sqrt{-1}\, g(R(\partial_i, Je_\alpha)e_\alpha, \partial_{\bar{j}})\right)$$
$$= \sum_\alpha g(R(\partial_i, \sqrt{2}\,\partial_{\bar{\alpha}})e_\alpha, \partial_{\bar{j}})$$
$$= \frac{1}{2}\left(\sum_\alpha \left(g(R(\partial_i, \sqrt{2}\partial_{\bar{\alpha}})e_\alpha, \partial_{\bar{j}}) - \sqrt{-1}\, g(R(\partial_i, \sqrt{2}\partial_{\bar{\alpha}})Je_\alpha, \partial_{\bar{j}})\right)\right)$$
$$= \frac{1}{2}\sum_\alpha g(R(\partial_i, \sqrt{2}\,\partial_{\bar{\alpha}})\sqrt{2}\,\partial_\alpha, \partial_{\bar{j}})$$
$$= \sum_\alpha R_{i\bar{\alpha}\alpha\bar{j}} = R_{i\bar{j}}. \quad \square$$

- Scalar curvature of Kähler metric ω is defined by

$$S(\omega) := g^{i\bar{j}} R_{i\bar{j}} = \frac{1}{2} R(g)$$

scalar curvature of the Riem metric g.