Lecture 7

Kähler Manifolds
Outline

1. Definition & examples
2. Basic properties
   - Volume form
3. Levi-Civita connection
4. Curvature
   - $H_K$
   - $B_K$
   - $R_a$
   - $R$

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Setup: Let $X$ be a complex manifold, and a complex structure $J$ of dim $n$. Here recall that $J^2 = -\text{id}$ and $J$ induces 3 operators on $X$ such that $d = 0 + S$. And $TX^c = TX^{1,0} \oplus TX^{0,1}$.

- Note that $X$ itself is also a diff. manifod so we may consider Riemannian metrics on $X$. A Riemannian metric $g$ on $X$ is called "Hermitian" if for all $x \in X, \forall u, v \in TX$, one has $g(Ju, Jv) = g(u, v)$, namely $g$ is $J$-invariant.

- Hermitian metrics always exist. This is because locally the standard Euclidean metric $g_e$ on $\mathbb{C}^n$ is Hermitian w.r.t. the standard complex structure. So we can use partition of unity to construct global Hermitian metrics on $X$. In what follows we will always assume that $g$ is Hermitian.

- Why the name "Hermitian"? This is because $g$ induces a Hermitian metric $h$ on the complexified $TX^c$ in the following way:
  - **Step 1.** Extend $g$ $\mathbb{C}$-linearly to $TX^c$.

  Then we find that
  \[
  \begin{align*}
  g(u, v) &= 0 \text{ for } \forall u, v \in TX^{1,0}, \\
  g(u, v) &= 0 \text{ for } \forall u, v \in TX^{0,1}.
  \end{align*}
  \]

  **Step 2.** Define $h$ by putting
  \[
  h(u, v) := g(u, \overline{v}) \text{ for } \forall u, v \in TX^c.
  \]

  Then $h$ is positive definite, $h(u, v) = \overline{h(v, u)}$.

  $Tu \in TX^c$ can be written as $u = Re u + Ji \text{ Im} u$.

  Then $g(Tu, T\overline{u}) = g(Re u + Ji \text{ Im} u, Re u - Ji \text{ Im} u)$

  \[
  = \|Re u\|^2 + \|\text{ Im} u\|^2.
  \]
Using $h$, one sees that $h(u,v) = 0$ for $u \in T_x \gamma, v \in T_y \gamma$. So we see that $h$ induces an orthogonal decomposition:

$$TX \gamma = T_x \gamma^0 \oplus T_x \gamma^1 \, \forall \, T_x \gamma \subset \gamma.$$ 

- The Kähler form (fundamental form) of $(X, g, J)$ is defined by $\omega := g(J \cdot , \cdot )$. We will see below that $\omega$ is a positive real $(1, 1)$-form.

- Now let us compute everything locally. Choose a hole coord. $(z^i, \ldots, z^n)$ write $z^i = x^i + x^j y^j$. Assume that $g$ is given by

$$g = g_{ij} dx^i \otimes dx^j + g_{ij} dy^i \otimes dy^j + g_{ij} dy^i \otimes dx^j + g_{ij} - dy^i \otimes dy^j$$

Then both $(g_{ij})$ and $(g_{ij})^*$ are symmetry & positive definite & $g_{ij} = g_{ji}^*$. Recall that $J dx^i = -dy^j \, \& \, J dy^i = dx^i$. So $g(J^* J) = g(J^*)$ implies that

$$g = g_{ij} dy^i \otimes dy^j - g_{ij} dy^i \otimes dx^j - g_{ij} dx^i \otimes dy^j + g_{ij} dx^i \otimes dx^j$$

Thus \[ g_{ij} = g_{ji}^* \Rightarrow (g_{ij}) \text{ is skew-symmetry.} \]

$$g = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$

So in conclusion $(g_{ij}) = (g_{ij})^* > 0 \, \& \, (g_{ij})^* = - (g_{ij}^*) = - (g_{ji})^T$.

Now we write out everything using $d\bar{z}^i$ & $dz^j$.

Plugging in $dz^i = \frac{1}{2} (dz^i + d\bar{z}^i)$ & $d\bar{z}^i = -\frac{1}{2} (dz^i - d\bar{z}^i)$ in $g$ we find

$$g = \frac{1}{2} \left[ g_{ij} (dz^i + d\bar{z}^j) \otimes (dz^i + d\bar{z}^j) - g_{ij} (dz^i - d\bar{z}^i) \otimes (dz^j - d\bar{z}^j) \right]$$

$$\Rightarrow h(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}) = \frac{1}{2} (g_{ij} + g_{ij}^*) \, \& \, h(\frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial z^j}) = \frac{1}{2} (g_{ij} - g_{ij}^*)$$
So as a Hermitian inner product on $T_X^C$, $h$ is given by

$$h = \frac{1}{2} \left( \begin{array}{cc} H & 0 \\ 0 & H \end{array} \right)$$

wrt. the basis $\{ \frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^j} \}$. 

We further compute $\omega$. Using $\int dx^i = \int dz^i$ & $\int d\bar{z}^i = -\int dz^i$.

$$\omega = h(J; i) = \frac{1}{2} (g_{ij} + ig_{ij}) dz^i \otimes d\bar{z}^j - \frac{1}{2} (g_{ij} - ig_{ij}) d\bar{z}^i \otimes dz^j$$

This computation shows that $\omega$ is real, positive of type $(1,1)$. 

Put $h_{ij} = g_{ij} + ig_{ij}$. Then $H = (h_{ij})$ is positive Hermitian.

Note that one can diagonalize $h_{ij}$ after a linear transformation of $z^1, \ldots, z^n$. In this case one has

$g_{ij} = \lambda_i$ & $g_{ij} = 0$ So $g$ is also diagonalized.

$$G = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \quad H = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$ 

Thus we find that $\det H = \sqrt{\det G}$. This holds everywhere.

One can also see this by diagonalization, as $G = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$ \( H = A + iB \). So $\det G = (\det(A + iB))^2 = (\det H)^2$.

\[ \text{Show that } \int \det G \ dx^i \otimes dy^i \wedge \cdots \wedge dx^n \otimes dy^n = \frac{\omega^n}{n!} \]

So in particular, $\omega^n$ defines a volume form on $X$.

After a coord. change, we have $g = \sum dx^i \otimes dx^i + \sum dy^j \otimes dy^j$.

$$x \omega = \frac{1}{2} \sum dx^i \otimes dx^i \Rightarrow \omega^n = n! \left( \frac{1}{2} dx^i \otimes dx^i \right) \wedge \cdots \wedge \left( \frac{1}{2} dx^n \otimes dx^n \right)$$

$$= n! \ dx^n \otimes dy^n \wedge \cdots \wedge dx^n \otimes dy^n.$$ 

Since $\omega^n$ is a volume form, it is also a Hermitian metric for $-K_X$.

Then the Chern curvature $-ig_{ij} \log \det \omega \in 2\pi C_1(X)$.

So one can compute $C_1(X)$ using Kähler form $\omega$ induced from $(X, g, J)$.

* The above discussion holds for all cplx mfds.
\* Def. \((X, g, J)\) is called \textbf{Kähler} if \(\text{dw} = 0\), where \(\text{w} = g(J)\).

\(\circ\) For simplicity, a Kähler mfd \(X\) will often be denoted by \((X, g)\) when the underlying \textit{cplx} struc \(J\) is fixed, in this setting \(J = \text{w}(J)\).

\(\circ\) A \textit{cplx} mfd is called \textbf{Kähler} if \(J\) Kähler metric \(g\).

Kähler condition is quite special. Although a \textit{cplx} mfd is always "locally Kähler", one cannot path these local Kähler metrics using partition of unity, as the \textit{d}-closedness is not likely to be preserved.

Examples

\(\circ\) \(\mathbb{C}^n\) equipped w/ the \textit{Euclidean} metric \(J = \mathbb{C}^n\) is Kähler.

in this case \(\text{w} = \tfrac{1}{2}(dz^1 \wedge d\overline{z}^1 + \ldots + dz^n \wedge d\overline{z}^n)\),

so clearly \(\text{dw} = 0\).

\(\circ\) \(B, \mathbb{C}^n\) is Kähler since \(\text{w} = 1/2 \log \frac{1}{\text{w}^{1/2}} > 0\). complex hyperbolic metric.

\(\circ\) \(J\) cplx submfd of a Kähler mfd is Kähler.

Assume that \(L\) is a \textit{holo.} line bundle on a \textit{cplx} mfd.

Assume that \(L\) admits a Hermitian metric \(h\) s.t. its
Chern curvature form \(R_h := -i\alpha \log h\) is positive definite,
meaning that, writing \(R_h = \frac{1}{2} A_{ij} dz^i \wedge d\overline{z}^j\) locally,
\((A_{ij})\) is a positive definite \textit{Hermitian} matrix.

In this case, \(R_h\) defines a Kähler metric \(g := R_h(J, J)\).

So \(X\) is Kähler. (We will see that \(X\) is actually projective.)

\(\circ\) \(\mathbb{C}P^n\) is Kähler.

To see this, we consider \(L := \mathcal{O}(1)\) on \(X\). Then we show that \(h := \frac{1}{(12z^1 + \ldots + 12z^n)}\), as a Hermitian metric on \(L\)
gives rise to a Kähler metric on \(\mathbb{C}P^n\).

On \(U_0 := \{ z_0 \neq 0\}\), \(R_h = i\alpha \log (1 + |z|^2 + \ldots + 13z_t^2)\),
here \(z_i = \frac{2iz_0}{z_0}\). We claim that \(R_h > 0\).
To see this, we use the fact that \( \mathbb{C}P^n \) is homogeneous, so we can assume that we are computing at \([1:0: \ldots :0]\).

At this point, \( R_n = \Sigma_i (d^3, \ldots d^3) > 0 \).
The induced Kähler metric has a special name, which is called Fubini–Study metric. \( R_n \) is denoted as \( \omega_\mathbb{S} \).

6. A complex submanifold of \( \mathbb{C}P^n \) is Kähler.

So in particular, a projective manifold is Kähler.

7. \( S^{2n} \) is NOT Kähler where \( n \geq 2 \).

Assume otherwise, \( S^{2n} \) admits a \( d \)-closed Kähler form \( \omega \).

Then it induces an element \([\omega] \in H^2_{\text{dR}}(S^{2n}, 0)\).

Since \( H^2_{\text{dR}}(S^{2n}, 0) \cong H^2(S^{2n}, 0) = 0 \) for \( n \geq 2 \), we see that \( \omega = d\theta \) for some 1-form \( \theta \) on \( S^{2n} \).

Thus \( \omega^n = d\theta \wedge \cdots \wedge d\theta = d(\theta \wedge d\theta \wedge \cdots \wedge d\theta) \).

So \( \int_{S^{2n}} \omega^n = \int_{S^{2n}} d(\theta \wedge d\theta \wedge \cdots \wedge d\theta) = 0 \) by Stokes' theorem, which is impossible since \( \omega^n \) is a volume form.

\( \blacklozenge \) For the same reason, Hopf surface \( X \cong S^1 \times S^3 \) is not Kähler.

\( \blacklozenge \) Exam. If \( X \) is a compact Kähler manifold, then \( H^2_i(X, \mathbb{R}) \neq 0 \) for all \( i \geq 0 \).

Pf. \( H^0(X, \mathbb{R}) = 0 \) clearly. For \( i \geq 1 \), consider \( \omega^i \), then \( d\omega^i = 0 \).

If \( \omega^i = d\theta \) for some \((2i-1)\)-form, then \( \omega^{2i} = d(\theta \wedge \omega^{2i-1}) \).

So \( \int_X \omega^{2i} = 0 \), which is absurd.

\( \blacklozenge \) We will see, using Hodge theory, that \( \dim H^{2i+1}(X, \mathbb{C}) \) is always even.

So being Kähler is a very restrictive condition.

In some sense, Kähler is very close to being "algebraic".
In what follows we fix a Kähler mfd $(X, g, J)$ whose associated Kähler form is denoted by $\Omega$. Let $\nabla$ denote the Levi-Civita connection of $g$. One can extend $\nabla$ by $\mathbb{C}$-linearity so that it defines a connection on $TX \otimes \mathbb{C}$.

We put for simplicity $\partial_i := \frac{\partial}{\partial z_i}$ and $\bar{\partial}_i := \frac{\partial}{\partial \bar{z}_i}$. Then $\nabla$ is determined by

$$\begin{align*}
\nabla_{\partial_i} \partial_j &= \Gamma^k_{ij} \partial_k + \Gamma^\bar{k}_{ij} \partial_{\bar{k}} \\
\nabla_{\partial_i} \bar{\partial}_j &= \bar{\Gamma}^k_{ij} \partial_k + \bar{\Gamma}^\bar{k}_{ij} \partial_{\bar{k}} \\
\nabla_{\partial_i} \bar{\partial}_j &= \bar{\Gamma}^k_{ij} \partial_k + \bar{\Gamma}^\bar{k}_{ij} \partial_{\bar{k}} \\
\nabla_{\partial_i} \partial_{\bar{j}} &= \Gamma^k_{ij} \partial_k + \Gamma^\bar{k}_{ij} \partial_{\bar{k}}.
\end{align*}$$

Since $\nabla$ is real, $
\Gamma^k_{ij} = \Gamma^\bar{k}_{ij}, \ \bar{\Gamma}^k_{ij} = \bar{\Gamma}^\bar{k}_{ij}, \ \bar{\Gamma}^\bar{k}_{ij} = \bar{\Gamma}^k_{ij} \ \& \ \bar{\Gamma}^{\bar{k}}_{ij} = \bar{\Gamma}^{\bar{k}}_{ij}.$

So it is enough to consider $\Gamma^k_{ij}, \bar{\Gamma}^{\bar{k}}_{ij}, \bar{\Gamma}^{\bar{k}}_{ij} \& \bar{\Gamma}^k_{ij}$.

Since $\nabla$ is torsion free, we have $\Gamma^k_{ij} = \Gamma^k_{ji}, \ \bar{\Gamma}^{\bar{k}}_{ij} = \bar{\Gamma}^{\bar{k}}_{ji}$.

We write $\omega = \omega_i \omega^j dz^i \wedge d\bar{z}^j$. Then $\left( \Gamma^k_{ij} = \bar{\Gamma}^{\bar{k}}_{ij}, \ \bar{\Gamma}^{\bar{k}}_{ij} = \bar{\Gamma}^k_{ij} \right)$.

Since $\nabla g = 0$, we have

$$0 = \partial_i \partial_j (\partial_k, \partial_{\bar{k}}) = g(\nabla_{\partial_i} \partial_k, \partial_{\bar{k}}) + g(\partial_k, \nabla_{\partial_i} \partial_{\bar{k}}),$$

$$= \Gamma^k_{ij} \omega^k \partial_{\bar{k}} + \Gamma^k_{ij} \omega^k \partial_k.$$  

Exchanging $i$ & $k$ we have $0 = \Gamma^k_{ij} \omega^k \partial_{\bar{k}} + \Gamma^k_{ij} \omega^k \partial_k$ as well.

Thus

$$\Gamma^k_{ij} \omega^k \partial_{\bar{k}} = \bar{\Gamma}^k_{ij} \omega^k \partial_{\bar{k}}.$$  

Thus

$$\Gamma^k_{ij} \omega^k \partial_k = \bar{\Gamma}^k_{ij} \omega^k \partial_k.$$  

Then

$$\Gamma^k_{ij} \omega^k \partial_{\bar{k}} = 0 \Rightarrow \Gamma^k_{ij} \omega^k = 0. \ \therefore \ \nabla_{\partial_i} \partial_j = \Gamma^k_{ij} \partial_k.$$  

On the other hand

$$\frac{\partial \omega^k}{\partial z_i} = \partial_i g(\partial_k, \partial_{\bar{k}}) = g(\nabla_{\partial_i} \partial_k, \partial_{\bar{k}}) + g(\partial_k, \nabla_{\partial_i} \partial_{\bar{k}})$$

$$= \Gamma^k_{ij} \omega^l \partial_{\bar{l}} + \Gamma^k_{ij} \omega^l \partial_{\bar{l}}.$$  

Exchanging $k \& \bar{k}$ we deduce that $\Gamma^k_{ij} \omega^k = \bar{\Gamma}^{\bar{k}}_{ij} \omega^k$.

Using $\partial_i g(\partial_i, \partial_k) = 0$ we find that $\Gamma^k_{ij} \omega^k = 0.$
Thus \( \Gamma_{ij}^k = 0 \) \( \Rightarrow \Gamma_{i}^{\frac{g}{k}} = \Gamma_{j}^{\frac{g}{k}} = 0 \) \( \Rightarrow \nabla_{\overline{j}} \overline{z}^i = 0 \).

So finally, we arrive at \( \partial \omega_{kij} = \Gamma_{kij} \). So the Levi-Civita connection of \( g \) is explicitly given by the condition
\[
\begin{align*}
\nabla_{\overline{j}} \overline{z}^i & = \omega_{kij} \frac{\partial \omega_{kij}}{\partial z^i} \overline{z}^j, \\
\nabla_{\overline{j}} \overline{z}^i & = 0.
\end{align*}
\]

So using the Kähler form \( (\omega = \sum W_{ij} \, dz^i \wedge d\overline{z}^j) \)
one can easily compute the Christoffel symbol.

This, again, shows that the geometry of a Kähler mfd
is completely determined by its Kähler form.

By abuse of language, we will also call \( \omega \)
"Kähler metric." And locally we will write
\[
\omega = \sum W_{ij} \, dz^i \wedge d\overline{z}^j.
\]

Then the corresponding Riem metric \( g \) is given by
\[
g = g_{ij} \, dz^i \otimes d\overline{z}^j + \overline{g}_{ij} \, d\overline{z}^i \otimes dz^j.
\]

\[\text{Lem. Let } (X, \omega) \text{ be a Kähler mfd. Then for } \forall p \in X,
\exists \text{ local hol. coord. } (z^1, \ldots, z^n) \text{ around } p \text{ s.t.}
\omega = i_X (\delta_{ij} + O(|z|^k)) \, dz^i \wedge d\overline{z}^j.
\]

**Proof:** First, one can choose \( (w^1, \ldots, w^n) \) around \( p \) s.t.
\[
\omega = \sum g_{ij} \, dw^i \wedge dw^j, \quad g_{ij} = \delta_{ij} + \frac{\partial g_{ij}}{\partial w^k}(p) \, w^k + \frac{\partial g_{ij}}{\partial w^k}(p) \overline{w}^k + \partial w^k.
\]

Put \( A_{ikj} := - \frac{\partial g_{ij}}{\partial w^k}(p). \) Then \( A_{ikj} = g_{kij}. \)

We set \( z^i := w^i - \frac{1}{2} A_{kij} \, w^k \). Then \( (z^1, \ldots, z^n) \) is
a hol. coord. s.t. \( w^i = z^i + \frac{1}{2} A_{kij} \overline{z}^k \overline{z}^j + O(|z|^2) \)
So that \( dw^i = dz^i + A_{kij} \overline{z}^k d\overline{z}^j \) \& \( d\overline{w}^j = d\overline{z}^j + A_{stj} \overline{z}^s d\overline{z}^t. \)

Plugging these into the expression of \( \omega, \)
\[ \omega = \sqrt{-1} \left( \delta_{ij} + \frac{\partial g_{ij}}{\partial x_k} \bar{z}^k + \frac{\partial g_{ij}}{\partial x_k} z^k + \tilde{c}(\bar{z}^l) \right) (d\bar{z}^i + \alpha_{ki} \bar{z}^k z^j) \left( d\bar{z}^j + \alpha_{kj} \bar{z}^j z^i \right) \]

\[ = \int (\delta_{ij} + \frac{\partial g_{ij}}{\partial x_k} \bar{z}^k + \frac{\partial g_{ij}}{\partial x_k} z^k + \tilde{c}(\bar{z}^l)) d\bar{z}^i \wedge d\bar{z}^j \]

\[ = \int (\delta_{ij} + \tilde{c}(\bar{z}^l)) d\bar{z}^i \wedge d\bar{z}^j . \]

\[\Box\]

**Cor.** Let \((X, \omega)\) be a Kähler manifold, then for every \(p \in X\)

\[ \exists \text{ holomorphic coord. } (z^1, \ldots, z^n) \text{ around } p \text{ s.t.} \]

\[ \nabla \bar{z}^i \bar{z}^j = 0 \text{ at } p. \]

Such coord. is called "Kähler normal coord. system".

\[\Box\]

In general this is not true for complex manifolds!

So this is another special property of Kähler manifolds.

**Cor.** If \((X, g, J)\) is Kähler, then \(\nabla J = 0\).

**pf:** In Kähler normal coord., \(J\) has constant coefficients:

\[ J = \prod_{i=1}^{n} \frac{\partial J}{\partial \bar{z}^i} \prod_{i=1}^{n} \frac{\partial J}{\partial z^i} \quad \text{(this holds in all holomorphic coord. system)} \]

\& \(\nabla J(p) = 0\). Thus \(\nabla J = 0\) everywhere. \(\Box\)

\[\Box\]

\(\nabla J = 0\) implies that \(\nabla(\bar{J}V) = J \nabla V\) for all real vector fields \(V\).

**Cor.** If \((X, g, J)\) is Kähler, then \(J R(U, V) J W = J R(U, V) W\).

\[ R(JU, JV) W = R(U, V) W . \]

**pf:**

\[ 0 (\nabla_U V - \nabla_V U - \nabla_U J V - \nabla_V J U) J W = J (\nabla_U V - \nabla_V U - \nabla_U J V - \nabla_V J U) W \]

\[ R(U, V) W = R(U, V) W . \]

\[\Box\]

\[\forall U, V, Z \quad J (R(JU, JV) W, Z) = R(JU, JV, Z, W) \text{, } J \text{ is Hermitian} \]

\[= R(Z, W, JU, JV) = R(Z, W, U, V) \]

\[= R(U, V, Z, W) = J (R(U, V) W, Z) . \]

\[\Box\]
Extending \( R \) \( C \)-linearity we get a 4-tensor on \( T^\infty \).

For any \( u, v \in T^\infty \) one has \( R(u, v) = R(u, iu) = -R(u, v) \).

So \( R(u, v) = 0 \). Similarly \( R(u, v) = 0 \) for any \( u, v \in T^\infty \).

Thus the only interesting component of \( R \) is

\[
R_{ijkl} := g \left( R(e_i, e_j) e_k, e_l \right)
\]

This convention is different from the Eilenberg case!

Assume

\[
\omega = \nabla g + d \omega
\]

\[
\omega = \frac{\partial P_i}{\partial x} \frac{g_{ij}}{\partial x} = -\left( g_{ij} \frac{\partial P_i}{\partial x} \right)
\]

\[
= - \frac{\partial^2 g_{ij}}{\partial x^k \partial x^l} + \frac{d g_{ij}}{d x^l} \frac{g_{ij}}{d x^k} \frac{d g_{ij}}{d x^l}
\]

\( R_{ijkl} \) satisfies \( R_{ijkl} = R_{klij} = R_{lijk} = R_{klij} \)

Define \( R_{ij} := g_{kl} R_{ijkl} \). Then \( \text{Ricci}(\omega) := \omega - R_{ij} d x^i d x^j \)

is called the Ricci form of \( \omega \).

One actually has \( R_{ij} = -g^{kl} \frac{\partial^2 g_{kl}}{\partial x^i \partial x^j} + g_{ij} \frac{\partial g_p}{\partial x^l} \frac{\partial^2 g_{kl}}{\partial x^p \partial x^l} \)

\[
= \frac{1}{2} \left( - g^{kl} \frac{\partial g_{kl}}{\partial x^j} \right)
\]

\[
= - \frac{\partial^2}{\partial x^i \partial x^j} \log \det (g_{kl})
\]

So we find that \( \text{Ricci}(\omega) = - \Delta i \partial^2 \log \det \omega \).

This in particular shows that \( \text{Ricci}(\omega) \) is the Chern curvature of the Hermitian metric \( \det \omega \) on \( Kx \).

So we have \( \text{[Ricci}(\omega)] = 2\pi C(X) \).
Thm. Let $\text{Ric}(\cdot, \cdot)$ denote the Ricci tensor of the Kähler metric $g$. Then

1. $\text{Ric}(J \cdot, J \cdot) = \text{Ric}(\cdot, \cdot)$
2. $\text{Ric}(w) = \text{Ric}(J \cdot, \cdot)$.

**Pf.** For all $X, Y \in \Gamma (X, TX)$, one has

\[ \text{Ric}(JX, JY) := g(\text{R}(JX, e_i) e_i, JY) = -g(\text{R}(JX, e_i) J Je_i, Y) = g(\text{R}(X, Je_i) Je_i, Y) = \text{Ric}(X, Y). \]

We compute using Kähler normal coord $(\bar{z}', \ldots, \bar{z}'' \cdot)$ around $p$. Also write $\bar{z}' = \bar{x}' + \bar{y}' i$. Then one has

\[ g = 2i \bar{d}x' \otimes dx' + 2 \bar{d}y' \otimes dy' \text{ at } p. \]

Write $e_i = \frac{1}{\sqrt{2}} \bar{d}x_i$ & $Je_i = \frac{1}{\sqrt{2}} dy_i$.

Then $\{ e_i, \ldots, e_n, Je_i, \ldots, Je_n \}$ is orthonormal at $p$.

Since $\text{Ric}(J \cdot, J \cdot) = \text{Ric}(\cdot, \cdot)$, one can write

\[ \text{Ric}(J \cdot, \cdot) = \sqrt{-1} \Theta_{ij} \bar{d}z^i \wedge \bar{d}z^j, \text{ then again using the local computation for } g, \text{ we have} \]

\[ \Theta_{ij} = \text{Ric}(\bar{z}' \cdot, \bar{z}' \cdot) = \sum_{\alpha} g(\text{R}(\bar{z}', Je_i) Je_i, \bar{z}' \cdot) + \sum_{\alpha} g(\text{R}(\bar{z}', e_\alpha) Je_i, \bar{z}' \cdot) \]

\[ = \sum_{\alpha} g(\text{R}(\bar{z}', Je_i) Je_i, \bar{z}' \cdot) + \sum_{\alpha} g(\text{R}(\bar{z}', e_\alpha) Je_i, \bar{z}' \cdot) \]

\[ = \sum_{\alpha} g(\text{R}(\bar{z}', e_\alpha) Je_i, \bar{z}' \cdot) \]

\[ = \frac{1}{2} \sum_{\alpha} \left( g(\text{R}(\bar{z}', e_\alpha) e_\alpha, \bar{z}' \cdot) - \bar{g}(\text{R}(\bar{z}', e_\alpha) Je_i, \bar{z}' \cdot) \right) \]

\[ = \frac{1}{2} \sum_{\alpha} g(\text{R}(\bar{z}', e_\alpha) e_\alpha, \bar{z}' \cdot) \]

\[ = \frac{1}{2} \sum_{\alpha} \text{Ric}(\bar{z}' e_\alpha, e_\alpha, \bar{z}' \cdot) = \text{Ric}(\bar{z}' \cdot, \bar{z}' \cdot). \]
• Scalar curvature of Kähler metric $\omega$ is defined by

$$S(\omega) := g^{i\overline{j}} R_{i\overline{j}} = \frac{1}{2} R(g)$$

\[\text{Scalar curvature of the Riemann metric } g.\]