Lecture 8

Hodge Theory on Complex Manifolds

$$
\begin{array}{ccccc}
 & & h^{0} & & \\
 & h^{1,0} & & h^{0,1} & \\
h^{2,0} & & h^{1,1} & & h^{0,2} \\
 & h^{2,1} & & h^{1,2} & \\
 & & h^{2,2} & &
\end{array}
$$

Outline

1. Hodge $*$ operator
2. Hodge theory on Riem mfd.
3. Hodge theory on complex mfd
4. Hodge theory on Kähler mfd.

- Let V be an $\mathbb{R}$-vector space of dim m w/ a Euclidean inner product $\langle\cdot,\cdot\rangle$. let $e^1,\cdots,e^m$ be an orthonormal basis. Fix $Vol = e^1\wedge\cdots\wedge e^m$. This gives an orientation of V.
  $\langle\cdot,\cdot\rangle$ can be extended to $\Lambda^k V$ for $\forall k\in\{1,\cdots,m\}$, whose ONB is just $\{e^{i_1}\wedge\cdots\wedge e^{i_k}\}_{1\le i_1<\cdots<i_k\le m}$.
  Now for $\forall$ $e^I\in\Lambda^k V$ (where $I=(i_1,\cdots,i_k)$, $i_1<\cdots<i_k$ is multi-index) define $*e^I := \mathrm{Sign}(I,I^c)\, e^{I^c}\in\Lambda^{m-k}V$.
  Extending $*$ linearly to $\Lambda^k V$ we thus obtain a linear map
  $$*:\Lambda^k V\longrightarrow \Lambda^{m-k}V.$$

e.x. ① Show that $\alpha\wedge*\beta = \langle\alpha,\beta\rangle\cdot Vol$ for $\forall\,\alpha,\beta\in\Lambda^k V$.

② show that $**=(-1)^{k(m-k)}$. So $*$ induces an isomorphism.

③ show that $\langle *\alpha,*\beta\rangle=\langle\alpha,\beta\rangle$. Thus $*$ induces an isometry.

- We further assume that V admits a complex structure J s.t. $\langle J\cdot,J\cdot\rangle=\langle\cdot,\cdot\rangle$. Then $m=2n$ must be even & $\langle\cdot,\cdot\rangle$ induces an Hermitian inner product $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ on $V^{\mathbb{C}}$ and
  $$\Lambda^k V^{\mathbb{C}} = \bigoplus_{p+q=k}\Lambda^{p,q}V^{\mathbb{C}}$$
  $\langle\alpha,\beta\rangle_{\mathbb{C}} := \langle\alpha,\bar\beta\rangle$ for $\forall\alpha,\beta\in\Lambda^k_{\mathbb{C}}$

If $\{e^1,\cdots,e^n,Je^1,\cdots,Je^n\}$ is ONB of $\langle\cdot,\cdot\rangle$ then $\{\frac{1}{\sqrt2}(e^i-\sqrt{-1}Je^i), \frac{1}{\sqrt2}(e^i+\sqrt{-1}Je^i)\}$ is ONB of $\langle\cdot,\cdot\rangle_{\mathbb{C}}$.

e.x. ④ Show that the above decomposition is orthogonal w.r.t. $\langle\cdot,\cdot\rangle_{\mathbb{C}}$.

$*$ can be extended to $V^{\mathbb{C}}$ as well. So one has

$$*: \Lambda^k V^{\mathbb{C}} \longrightarrow \Lambda^{2n-k} V^{\mathbb{C}}.$$

e.x. ⑤ show that $\alpha \wedge * \bar{\beta} = \langle \alpha, \beta \rangle_{\mathbb{C}} \cdot vol$ for $\forall \alpha, \beta \in \Lambda^k V^{\mathbb{C}}$.

⑥ If $\alpha \in \Lambda^{p,q} V^{\mathbb{C}}$, then $*\alpha \in \Lambda^{n-q, n-p} V^{\mathbb{C}}$.
So $*$ induces an isomorphism: $\Lambda^{p,q} V^{\mathbb{C}} \longrightarrow \Lambda^{n-q,n-p} V^{\mathbb{C}}$.

Write $*\alpha = \sum_{r+s=2n-p-q} \alpha^{r,s}$. Using the fact that $\gamma^{q,p} \wedge \alpha^{r,s} = 0$ whenever $q+r \neq n$ or $p+s \neq n$ & the equality $\gamma \wedge * \alpha = \langle \gamma, \bar{\alpha} \rangle_{\mathbb{C}}$ we find that all $\alpha^{r,s}$ w/ $(r,s) \neq (n-q, n-p)$ are zero.

⑦ $\langle *\alpha, *\beta \rangle_{\mathbb{C}} = \langle \alpha, \beta \rangle_{\mathbb{C}}$ for $\forall \alpha, \beta \in \Lambda^{p,q} V^{\mathbb{C}}$.

$$\langle *\alpha, *\beta \rangle_{\mathbb{C}} = \langle *\alpha, \overline{*\beta} \rangle = \langle \overline{*\beta}, *\alpha \rangle = (-1)^{(p+q)(2n-p-q)} \overline{*\beta} \wedge \alpha$$

$$= \alpha \wedge \overline{*\beta} = \overline{\bar{\alpha} \wedge *\beta} = \overline{\langle \bar{\alpha}, \beta \rangle}$$

$$= \langle \alpha, \bar{\beta} \rangle = \langle \alpha, \beta \rangle_{\mathbb{C}}.$$

So $*$ induces an isometry $\Lambda^{p,q} \longrightarrow \Lambda^{n-q, n-p}$.

- Now let (X, g) be a cpt m-dim oriented Riem mfd. Consider $\Lambda^k T^* X$.
  We put $A^k(X, \mathbb{R}) := C^\infty(X, \Lambda^k T^* X) = \{C^\infty, \mathbb{R}\text{-valued global } k\text{-forms on } X\}$
  Then the above point wise discuss yields a globally defined operator
  $$*: A^k(X, \mathbb{R}) \longrightarrow A^{m-k}(X, \mathbb{R}), \quad 1 \leq k \leq n.$$
  This is called the Hodge star operator of (X, g).

★ Note that $*$ dpd on g.
For $\forall \alpha, \beta$ k-forms, one has $\alpha \wedge * \beta = g(\alpha, \beta)\, dV_g$
where $dV_g := \sqrt{\det g}\, dx^1 \wedge \cdots \wedge dx^n \wedge dy^1 \wedge \cdots \wedge dy^n$.

- Define $(\cdot, \cdot)$ on $A^k(X, \mathbb{R})$ by $(\alpha, \beta) := \int_X g(\alpha, \beta)\, dV_g$.

- Define $d^*: A^k(X, \mathbb{R}) \to A^{k-1}(X, \mathbb{R})$ by putting
  $$d^* := (-1)^{mk+m+1} * d * \qquad (d^{*2} = 0)$$

- **Prop.** For $\forall\ \alpha \in A^k(X)$, $\beta \in A^{k-1}(X)$, one has
$$(\alpha, d\beta) = (d^*\alpha, \beta).$$ So d^* is the adjoint operator of d w.r.t. $(\cdot,\cdot)$.

pf: $$(d^*\alpha, \beta) = \int_X g(d^*\alpha, \beta)\, dV_g = \int_X g(\beta, d^*\alpha)\, dV_g$$
$$= \int_X \beta \wedge * d^*\alpha = (-1)^{mk+m+1} \int_X \beta \wedge * * \underbrace{d * \alpha}_{m-k+1}$$
$$= (-1)^{mk+m+1+(m-k+1)(k-1)} \int_X \beta \wedge d * \alpha$$
$$= (-1)^{k^2} \int_X \beta \wedge d * \alpha = (-1)^{k^2+k-1} \int_X d(\beta \wedge *\alpha) - d\beta \wedge *\alpha$$
$$\overset{\text{Stokes}}{=} \int_X d\beta \wedge * \alpha = \int_X g(d\beta, \alpha)\, dV_g$$
$$= \int_X g(\alpha, d\beta)\, dV_g = (\alpha, d\beta). \quad \square$$

- **Rmk.** Using the Levi-Civita connection ∇ of g, one also has
$$\nabla : A^k(X, \mathbb{R}) \to A^{k+1}(X, \mathbb{R}).$$
One can also define the adjoint ∇^*, which satisfies
$$(\alpha, \nabla\beta) = (\nabla^*\alpha, \beta)$$

Q. What is ∇^* when acting on 1-forms?

- **Def.** We say $\alpha \in A^p(X)$ is a harmonic p-form if
$$d\alpha = 0 \ \& \ d^*\alpha = 0.$$

▲ If $\alpha \in A^p(X)$ is harmonic, then it defines an element $\xi \in H^p_{dR}(X, \mathbb{R})$.
We claim that α minimises the L^2 norm (β, β) for $\beta \in \xi$.
Indeed, $\forall\ \beta = \alpha + d\eta$, one has
$$(\beta, \beta) = (\alpha + d\eta, \alpha + d\eta) = (\alpha, \alpha) + (d\eta, d\eta) + 2(\alpha, d\eta)$$
$$= (\alpha, \alpha) + (d\eta, d\eta) \geq (\alpha, \alpha).$$
Thus α is the unique minimizer.

▲ Conversely, for $\forall\, \xi \in H^p_{dR}(X,\mathbb{R})$, if $\exists\, \alpha \in \xi$ s.t. α is a minimizer, then α has to be harmonic. Indeed, consider $\alpha_t = \alpha + t\,d\eta$ for $\forall\, \eta \in A^{p-1}(X,\mathbb{R})$.

$$0 = \frac{d}{dt}\Big|_{t=0} (\alpha_t, \alpha_t) = 2(\alpha, d\eta) = 2(d^*\alpha, \eta).$$

So $(d^*\alpha, \eta) = 0$ for $\forall\, \eta \in A^{p-1}(X,\mathbb{R})$.
Thus $d^*\alpha = 0$. Since α is d-closed already, so α is harmonic p-form.

- We put $\mathcal{H}^p(X,\mathbb{R}) := \{$ (real) harmonic p-forms on X $\}$.
- Define Hodge Laplace $\Delta := dd^* + d^*d$.
- Prop: $\alpha \in \mathcal{H}^p(X,\mathbb{R})$ iff $\Delta\alpha = 0$.

  pf: If $\Delta\alpha = 0$, then $0 = (\Delta\alpha, \alpha) = (d\alpha, d\alpha) + (d^*\alpha, d^*\alpha)$.
  So $d\alpha = 0$ & $d^*\alpha = 0$. $\square$.

★ • $\boxed{\text{Hodge Decomposition Thm.}}$

One has
$$A^p(X,\mathbb{R}) = \mathcal{H}^p(X,\mathbb{R}) \oplus d A^{p-1} \oplus d^* A^{p+1}$$
$$= \operatorname{Ker} d \oplus d^* A^{p+1}.$$

This decomposition is orthogonal w.r.t. $(\cdot,\cdot)$.

- As a consequence one has

$$\boxed{\mathcal{H}^p(X,\mathbb{R}) \cong H^p_{dR}(X,\mathbb{R})}$$
$$\alpha \longmapsto [\alpha].$$

Namely, each $\xi \in H^p_{dR}(X,\mathbb{R})$ admits a unique harmonic representative $\alpha \in \xi$ which minimizes the energy $\|\beta\|^2 := (\beta,\beta)$ for $\beta \in \xi$.

- ex. 8 Check that $*\Delta = \Delta *$.
  This implies that $*: \mathcal{H}^p(X,\mathbb{R}) \to \mathcal{H}^{m-p}(X,\mathbb{R})$
  is an isomorphism.

- As a consequence, we find that the pairing

$$H^p_{dR}(X,\mathbb{R}) \times H^{m-p}_{dR}(X,\mathbb{R}) \to \mathbb{R}$$
$$[\alpha] \quad , [\beta] \longmapsto \int_X \alpha \wedge \beta$$

  is non-degenerate.
  In fact, $\forall$ ONB $\{\alpha_i\}$ of $\mathcal{H}^p(X,\mathbb{R})$ gives an ONB $\{*\alpha_i\}$ of $\mathcal{H}^{m-p}(X,\mathbb{R})$ & we have

$$\int_X \alpha_i \wedge * \alpha_j = \int_X g(\alpha_i,\alpha_j)\, dV_g = \delta_{ij}.$$

  So we recover the "Poincare duality" in a very computable way.

- From now on, assume that X is a cplx mfd, cpt, of dim n, w/ a Hermitian metric g. We will extend everything discussed above $\mathbb{C}$-linearly to $A^k(X,\mathbb{C})$.
  Then $A^k(X,\mathbb{C}) = \bigoplus_{p+q=k} A^{p,q}(X)$.
  Also put $(\cdot,\cdot)_{\mathbb{C}} := \int_X h(\cdot,\cdot)\, dV_g$, where h is the Hermitian metric induced by g so that $h(\alpha,\beta) := g(\alpha,\bar{\beta})$ for $\forall \alpha,\beta \in A^k(X,\mathbb{C})$.
  Recall that (e.x. 6) $*: A^{p,q} \xrightarrow{\sim} A^{n-q,n-p}$
  Also note that $d^* = - * d *$ in this case as $\dim_{\mathbb{R}} X$ is even.
  But note that in general it makes no sense to talk about "harmonic (p,q)-forms" using the Hodge Laplace, since in general $\Delta\alpha$ is no longer of type (p,q) even if α is.
  So in general one cannot decompose $\mathcal{H}^k$ into $\mathcal{H}^{p,q}$.
  (But this is indeed true when X is Kähler)
  In the complex setting what we do instead is to decompose
  $d^* = \partial^* + \bar{\partial}^*$, where

$\bar\partial^{*2} = \partial^{*2} = 0$

$$\begin{cases} \bar\partial^* := -*\partial* : A^{p,q} \to A^{p,q-1} \\ \partial^* := -*\bar\partial* : A^{p,q} \to A^{p-1,q}. \end{cases}$$

- Prop. For $\forall\, \alpha \in A^{p,q}$, $\beta \in A^{p,q-1}$, one has
$$(\alpha, \bar\partial\beta)_{\mathbb{C}} = (\bar\partial^*\alpha, \beta)_{\mathbb{C}}.$$

Pf: $(\bar\partial^*\alpha, \beta)_{\mathbb{C}} = \int_X h(\bar\partial^*\alpha, \beta)\, dV_g$

$= \int_X g(\bar\beta, \bar\partial^*\alpha)\, dV_g$

$= \int_X \bar\beta \wedge * \underbrace{\bar\partial^*\alpha}_{p+q-1} = (-1)^{p+q} \int_X \bar\beta \wedge \partial * \alpha$

$= \int_X \partial\bar\beta \wedge *\alpha - \int_X \partial(\underset{(q,p-1)}{\bar\beta} \wedge \underset{(n-q,n-p)}{*\alpha})$

$= \int_X \partial\bar\beta \wedge *\alpha - \int_X d(\bar\beta \wedge *\alpha)$

$= \int_X g(\partial\bar\beta, \alpha)\, dV_g = \int_X g(\alpha, \overline{\bar\partial\beta})\, dV_g$

$= (\alpha, \bar\partial\beta)_{\mathbb{C}}.$ $\square$

- e.x. Show that $(\alpha, \partial\beta)_{\mathbb{C}} = (\partial^*\alpha, \beta)_{\mathbb{C}}$. for $\alpha \in A^{p,q}$, $\beta \in A^{p-1,q}$.

- Define. $\Delta_{\bar\partial} := \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ & $\Delta_\partial := \partial\partial^* + \partial^*\partial$.
  We say $\alpha \in A^k(X,\mathbb{C})$ is $\partial\,(\bar\partial)$-harmonic if $\Delta_\partial\alpha = 0\ (\Delta_{\bar\partial}\alpha = 0)$.

Then, α is ∂-harmonic iff $\partial\alpha = \partial^*\alpha = 0$
  α is $\bar\partial$-harmonic iff $\bar\partial\alpha = \bar\partial^*\alpha = 0$.

- Define $\mathcal{H}^k_{\bar\partial} := \{\alpha \in A^k(X,\mathbb{C}) \mid \Delta_{\bar\partial}\alpha = 0\}$.
$$\left\{ \mathcal{H}^{p,q}_{\bar\partial} := \{\alpha \in A^{p,q} \mid \Delta_{\bar\partial}\alpha = 0\}. \right.$$
Analogously, can define $\mathcal{H}^k_\partial$ & $\mathcal{H}^{p,q}_\partial$. All these dpd on g!

- $\boxed{\text{Hodge Decomposition II.}}$

① One has $\mathcal{H}_{\bar\partial}^{k} = \bigoplus_{p+q=k} \mathcal{H}_{\bar\partial}^{p,q}$ & $\mathcal{H}_{\partial}^{k} = \bigoplus \mathcal{H}_{\partial}^{p,q}$.

So $\alpha \in A^k$ is $\bar\partial$-harmonic iff each (p,q) piece is $\bar\partial$-harmonic.

② One has orthogonal decomposition

$$A^{p,q} = \mathcal{H}_{\bar\partial}^{p,q} \oplus \bar\partial A^{p,q-1} \oplus \bar\partial^* A^{p,q+1}$$

$$A^{p,q} = \mathcal{H}_{\partial}^{p,q} \oplus \partial A^{p-1,q} \oplus \partial^* A^{p+1,q}$$

③ $$\mathcal{H}_{\bar\partial}^{p,q} \cong H_{\bar\partial}^{p,q} := \frac{\operatorname{Ker}(\bar\partial : A^{p,q} \to A^{p,q+1})}{\operatorname{Im}(\bar\partial : A^{p,q-1} \to A^{p,q})}$$

$$\mathcal{H}_{\partial}^{p,q} \cong H_{\partial}^{p,q} := \frac{\operatorname{Ker}(\partial : A^{p,q} \to A^{p,q+1})}{\operatorname{Im}(\partial : A^{p,q-1} \to A^{p,q})}$$

- $*$ induces isometry: $\mathcal{H}_{\bar\partial}^{p,q} \xrightarrow[\sim]{*} \mathcal{H}_{\partial}^{n-q,n-p}$.

Also note that conjugation induces isomorphism:

$$\mathcal{H}_{\bar\partial}^{p,q} \cong \mathcal{H}_{\partial}^{q,p}.$$

Thus $\mathcal{H}_{\bar\partial}^{p,q}$ completely determines $\mathcal{H}_{\partial}^{p,q}$.

- The pairing $\mathcal{H}_{\bar\partial}^{p,q} \times \mathcal{H}_{\bar\partial}^{n-p,n-q} \to \mathbb{C}$

$\boxed{\text{Serre Duality}}$ $\qquad \alpha \quad , \quad \beta \longmapsto \int_X \alpha \wedge \beta$

is non-degenerate. Indeed, if $\alpha \in \mathcal{H}_{\bar\partial}^{p,q}$, then $*\bar\alpha \in \mathcal{H}_{\bar\partial}^{n-p,n-q}$ so that $\int_X \alpha \wedge *\bar\alpha = \int_X g(\alpha,\bar\alpha)\, dV_g \geq 0$.

So if $\{\alpha_i\}$ ONB of $\mathcal{H}_{\bar\partial}^{p,q}$, then $\{*\bar\alpha_i\}$ is ONB of $\mathcal{H}_{\bar\partial}^{n-p,n-q}$.

- As a consequence, $\dim H_{\bar\partial}^{p,q} = \dim H_{\partial}^{q,p} = \dim H_{\partial}^{n-q,n-p} = \dim H_{\bar\partial}^{n-p,n-q}$.

& $H^q(X, \Omega_X^p) \cong H^{n-q}(X, \Omega_X^{n-p})$ ← sheaf cohomology. (Ω_X^p: sheaf of holo. $(p,0)$ forms)

So $H^q(X, K_X) \cong H^{n-q}(X, \mathcal{O}_X)$.

★ Warning: in general it is not true that
$$\dim H_{\bar\partial}^{p,q} = \dim H_{\partial}^{q,p}\ !$$

★ Also warning: $\Delta \neq \Delta_\partial + \Delta_{\bar\partial}$ in general ($\partial\bar\partial^* + \bar\partial^*\partial \neq 0$ in general)

- We mention that for $\forall$ holo. v.b. E over a cpt cplx mfd X one has (after choosing Hermitian metrics) a Hodge theory for E-valued (p,q)-forms s.t.

$\mathcal{H}_{\bar\partial}^{p,q}(X,E) \cong H^q(X, \Omega_X^p \otimes E)$, and one has a general Serre duality
$$H^q(X, \Omega_X^p \otimes E) \cong H^{n-q}(X, \Omega_X^{n-p} \otimes E^*).$$

- In the Kähler setting, one has
$$\Delta = \Delta_\partial + \Delta_{\bar\partial} = 2\Delta_\partial = 2\Delta_{\bar\partial}.$$
So $\Delta_\partial = \Delta_{\bar\partial} = \frac{1}{2}\Delta$ are real operators.

These are proved using "Kähler identities". We omit the detail.

- Let (X,g) be a cpt Kähler mfd, then
$$\mathcal{H}^k(X,\mathbb{C}) = \mathcal{H}_{\bar\partial}^k = \mathcal{H}_\partial^k = \bigoplus_{p+q=k} \mathcal{H}_{\bar\partial}^{p,q} = \bigoplus_{p+q=k} \mathcal{H}_{\partial}^{p,q} \quad \text{w/ } \mathcal{H}_{\bar\partial}^{p,q} = \mathcal{H}_{\partial}^{p,q}.$$
($\mathcal{H}^k(X,\mathbb{C})$: $\mathbb{C}$-valued harmonic forms)

So one finds that $H^k(X,\mathbb{C}) \cong \bigoplus_{p+q=k} H_{\bar\partial}^{p,q}(X,\mathbb{C})$ ← This is indpd of g !! Check!

In this setting, $\forall$ $\mathbb{C}$-valued harmonic k-form can be decomposed as the sum of harmonic (p,q)-forms w/ $p+q=k$.

But such decomposition dpd on the Kähler metric.

However, $h^{p,q} := \dim H_{\bar\partial}^{p,q}$ is indpd of g & $b_k = \sum_{p+q=k} h^{p,q}$
($h^{p,q}$: Hodge number)

• **Cor**. The betti number b_{2k+1} of a cpt Kähler mfd must be even.

pf: this follows from $h^{p,q} = h^{q,p}$. $\square$.

☆ The Kähler form ω itself is a harmonic $(1,1)$ form!
(e.x. Show that $\bar{\partial}^* \omega = 0$)
This implies that $[\omega] \in H^2_{dR}(X,\mathbb{R}) \cap H^{1,1}(X,\mathbb{R})$ is non-trivial.
For $\forall$ Kähler form ω, we call $[\omega]$ the Kähler class of ω.
Conversely, for $\forall\ \xi \in H^2_{dR}(X,\mathbb{R}) \cap H^{1,1}(X,\mathbb{R})$, if $\exists$ a positive representative $\alpha \in \xi$, then α defines a Kähler metric.
The set $\{$ Kähler forms in $H^2_{dR}(X,\mathbb{R}) \cap H^{1,1}(X,\mathbb{R})\}$ is called the Kähler cone of X, denoted by $\mathcal{K}(X)$.

• e.x. 9 Assume that ω & ω' are two Kähler forms s.t. $[\omega] = [\omega']$. Then $\exists\ \varphi \in C^\infty(X,\mathbb{R})$ s.t.
$\omega' = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$.

$\boxed{\partial\bar{\partial}\text{-lemma}}$