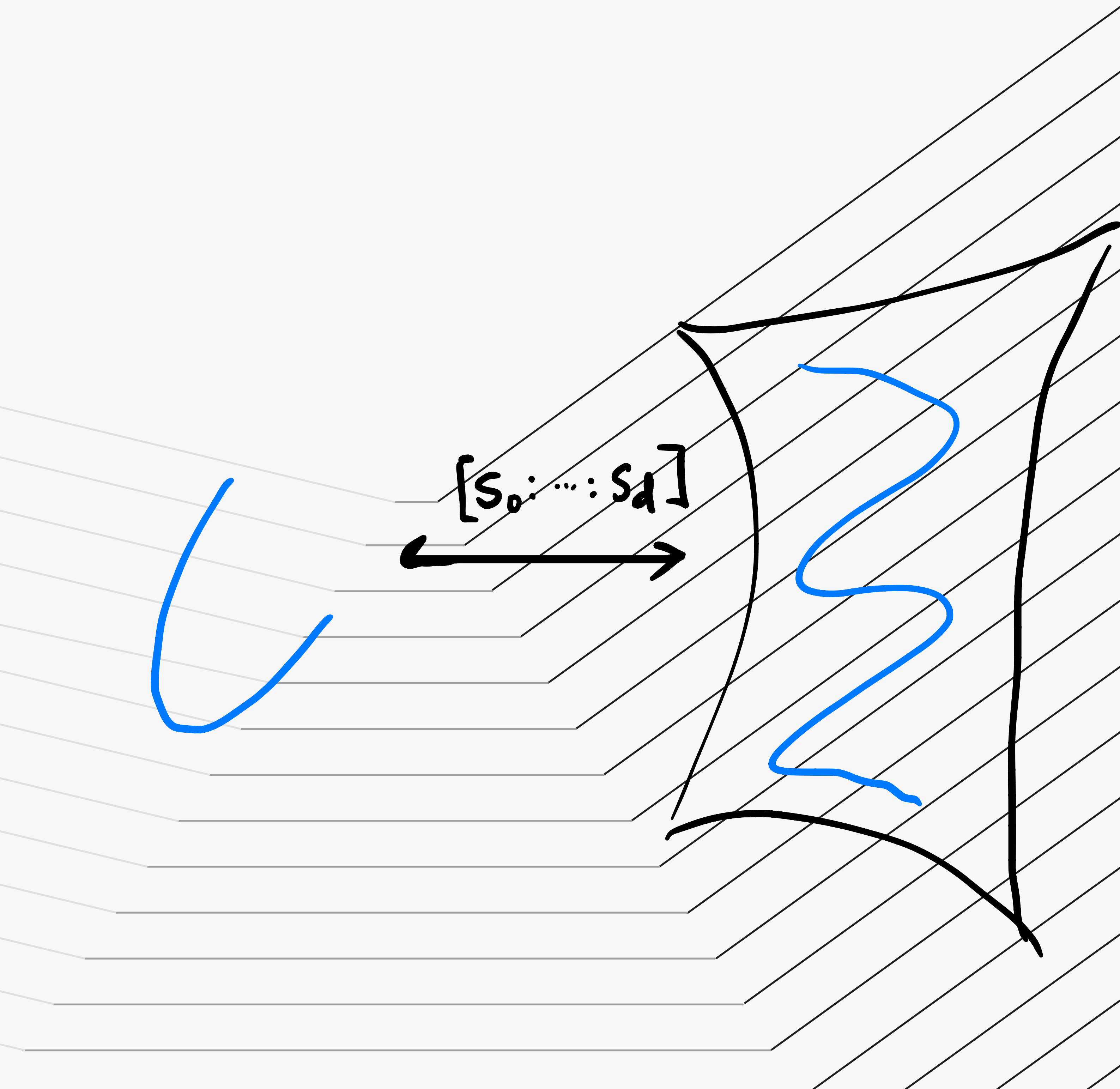
Lecture 9

Kodaira Embedding



Outline.

1. Lefschetz (1,1) theorem.
2. Ample line bundle & positive line bundle
3. Kodaira vanishing.
4. Kodaira embedding.

- Let X be a ~~cpt~~ cplx mfd w/ a holo. line bundle L on it.
  We have seen that, using the inclusion $\mathbb{Z} \to \mathbb{R}$, $c_1(L)$ can be represented by a closed real (1,1) form, a curvature form of L.
  Now we ask the following question: if $\xi \in H^2(X,\mathbb{R}) \cap \operatorname{Im}(H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{R}))$ can be represented by a closed real 2-form $\alpha \in \xi$ that is of type (1,1), then can one find a holo. l.b. L over X s.t. $c_1(L) = \xi$?
  To study this problem, we look at the exact sequence:

$$\operatorname{Pic}(X) \cong H^1(X, \mathcal{O}_X^*) \xrightarrow{c_1} H^2(X,\mathbb{Z}) \xrightarrow{\tau} H^2(X,\mathcal{O}_X) \cong H^{0,2}_{\bar\partial}(X).$$

We find that $\operatorname{Ker} \tau = \operatorname{Im} c_1$. So we need to study the map

$$H^2(X,\mathbb{Z}) \xrightarrow{\tau} H^2(X,\mathcal{O}_X)$$

In what follows, we will use a larger inclusion $\mathbb{Z} \hookrightarrow \mathbb{R} \hookrightarrow \mathbb{C}$.

$$\begin{tikzcd}
\mathbb{R} \arrow[r] \arrow[d, hook] & A^0_{\mathbb{R}} \arrow[r, "d"] \arrow[d, hook] & A^1_{\mathbb{R}} \arrow[r, "d"] \arrow[d, hook] & A^2_{\mathbb{R}} \arrow[r] \arrow[d, hook] & \cdots \\
\mathbb{C} \arrow[r] \arrow[d, hook] & A^0_{\mathbb{C}} \arrow[r, "d"] \arrow[d, equal] & A^1_{\mathbb{C}} \arrow[r, "d"] \arrow[d, "\pi^{0,1}"] & A^2_{\mathbb{C}} \arrow[r] \arrow[d, "\pi^{0,2}"] & \cdots \\
\mathcal{O}_X \arrow[r] & A^0_{\mathbb{C}} \arrow[r, "\bar\partial"] & A^{0,1} \arrow[r, "\bar\partial"] & A^{0,2} \arrow[r] & \cdots
\end{tikzcd}$$

e.x. This diagram is commutative.

From this we get an explicit description of $H^2(X,\mathbb{C}) \to H^2(X,\mathcal{O}_X)$: for $\forall$ $\mathbb{C}$-valued d-closed 2-form α, write $\alpha = \alpha^{0,2} + \alpha^{1,1} + \alpha^{2,0}$.
Then $\bar\partial \alpha^{0,2} = 0$. So $[\alpha^{0,2}]$ defines an element in $H^{0,2}_{\bar\partial}(X)$.
The map

$$\begin{aligned} H^2_{dR}(X,\mathbb{C}) &\to H^{0,2}_{\bar\partial}(X) \\ [\alpha] &\mapsto [\alpha^{0,2}] \end{aligned}$$

is well-defined.

Now the map $H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O}_X)$ can be described as follows:

Using Čech cohomology, $\forall [\alpha] \in H^2(X,\mathbb{Z})$ gives rise to a d-closed real 2-form on X, so treating it as a $\mathbb{C}$-valued 2-form one can take its $(0,2)$ part $\alpha^{0,2}$.
Then the map $H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O}_X)$ is given by $[\alpha] \mapsto [\alpha^{0,2}]$.
This actually holds for $H^{2k}(X,\mathbb{Z}) \to H^2(X,\mathcal{O}_X)$, $k \geq 1$

Now, what is the kernel of this map?
By the above discussion, $\forall\, \xi \in \mathrm{Im}(H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C}))$ is mapped to zero in $H^2(X,\mathcal{O}_X)$ iff $\exists$ (so any) representative $\alpha \in \xi$, its $(0,2)$ part $\alpha^{0,2}$ is $\bar\partial$-exact, hence $[\alpha^{0,2}]=0$ in $H^2(X,\mathcal{O}_X)$.
So we see that $\exists$ holo. l.b. L s.t. $c_1(L) = \xi \in \mathrm{Im}(H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{R}))$ iff $\exists$ (so any) representative $\alpha \in \xi$ satisfies that $\alpha^{0,2}$ is $\bar\partial$-exact.

- **Cor.** A class $\xi \in \mathrm{Im}(H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{R}))$ is the 1st Chern class of a holo. line bundle iff one can find a representative $\alpha \in \xi$ s.t. α is of type $(1,1)$.

pf: If $\xi = c_1(L)$ for some holo. L, then the Chern curvature form (normalized by $\frac{i}{2\pi}$) is of type $(1,1)$, representing ξ.
Conversely, if $\exists$ $(1,1)$ representative in ξ, we see that ξ lies in the kernel of $H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O}_X)$,
So $\exists$ holo. L s.t. $c_1(L) = \xi$ (modulo torsion of course) $\square$.

See [Huybrechts's book Thm 2.6.26]

- Another way to prove this is to use the fact that $\forall$ element $\xi \in \mathrm{Im}(H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{R}))$ corresp. to a cplx line bundle L, so the goal is to find a "holomorphic structure on L", i.e., a "$\boxed{\bar\partial_L\text{-operator}}$" $\mathcal{A}^0(L) \to \mathcal{A}^{0,1}(L)$ that satisfies the Leibnitz rules and $\bar\partial_L^2 = 0$. This is where the type $(1,1)$ condition comes in: first, pick $\forall$ cplx connection ∇ on L, then its curvature s.t. $\nabla^2 = \omega + d\alpha$ for some 1-form α.
Define $\bar\partial_L$ to be the $(0,1)$ part of $\nabla + \alpha$. Then we check that $\bar\partial_L^2 = 0$ since $(D+\alpha)^2 = \omega$ has no type $(0,2)$ part. $\square$.

- In the cpt Kähler setting, the above result is known as $\boxed{\text{Lefschetz theorem on (1,1)-classes}}$, which is stated as follows: let X be cpt Kähler.

  (1,1) piece of Hodge dcp.

  $\boxed{\text{Define } H^{1,1}(X,\mathbb{Z}) := \mathrm{Im}(H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C})) \cap H^{1,1}_{\bar\partial}(X) \text{. Then the map } \mathrm{Pic}(X) \to H^{1,1}_{\bar\partial}(X,\mathbb{Z}) \text{ is surjective.}}$

  pf: We go back to the proof the previous lemma.
  In the Kähler setting $H^2(X,\mathbb{C}) = H^{2,0}_{\bar\partial}(X) \oplus H^{1,1}_{\bar\partial}(X) \oplus H^{0,2}_{\bar\partial}(X)$
  for $\forall\, \xi \in \mathrm{Im}(H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C}))$, we can find a harmonic representative $\alpha \in \xi$ (w.r.t. some Kähler metric).
  Since $\xi \in H^2(X,\mathbb{R}) \subseteq H^2(X,\mathbb{C})$, α is real.
  Then $\overline{\alpha^{2,0}} = \alpha^{0,2}$. Now using the exact sequence

  $$\cdots \to \mathrm{Pic}(X) \xrightarrow{c_1} H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O}_X) \to \cdots$$

  ξ is of the form $c_1(L)$ iff $\alpha^{0,2} = 0$ so that $\alpha = \alpha^{1,1}$. $\square$

- Def. A holo. line bundle L over X is called $\boxed{\text{positive}}$ if $\exists$ Hermitian metric h on L s.t. its curvature form R_h is positive definite (so it gives a Kähler metric).

- Prop: A cplx mfd admits a positive line bundle iff $\exists\, \xi \in \mathrm{Im}(H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{R}))$ s.t. $\exists$ Kähler form $\omega \in \xi$.
  pf: This is clear from the above discussions. $\square$

  The following result is also clear.

- Prop: A cpt Kähler mfd X admits a positive line bundle iff $H^{1,1}(X,\mathbb{Z}) \cap \underbrace{K(X)}_{\text{Kähler cone}} \neq \emptyset$

- Prop. Let X be cpt Kähler. If $h^{0,2} = 0$, then X admits a positive line bundle.

pf: $h^{0,2} = h^{2,0} = 0$ implies that $\dim H^2(X,\mathbb{R}) = h^{1,1}$, so $\mathrm{Im}(H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{R}))$ spans the entire $H^2(X,\mathbb{R})$.
Thus $\exists$ holo. line bundles $L_1, \cdots, L_b$ & $a_1, \cdots, a_b \in \mathbb{R}$ s.t.
$a_1 c_1(L_1) + \cdots + a_b c_1(L_b) = [\omega]$.
Slightly perturbing ω & a_i, we may assume that $a_i \in \mathbb{Q}$.
Kähler cone is open!
Then a sufficiently divisible integer N w/ $Na_i \in \mathbb{Z}$ defines a holo. line bundle $L := Na_1 L_1 + \cdots + Na_b L_b$ s.t. $c_1(L) = [N\omega]$ is represented by a Kähler form. $\square$.

• **Def.** A holo. line bundle L over a cpt cplx mfd is called $\boxed{\text{ample}}$ if for $\forall k \gg 0$, a basis $\{s_0, \cdots, s_{d_k}\}$ of $H^0(X, kL)$ defines an embedding:

$$X \xrightarrow{i} \mathbb{CP}^{d_k}$$
$$x \longmapsto [s_0(x) : \cdots : s_{d_k}(x)].$$

For such k, kL is called $\boxed{\text{very ample}}$.
Check that in this case $kL = i^* \mathcal{O}(1)$.

• It is clear from the definition that

Prop. A cpt cplx mfd is projective iff it admits an ample line bundle.

pf: One direction is direct from the definition.
For the other direction, if X embedds in $\mathbb{CP}^N$, then $\mathcal{O}(1)|_X$ is ample.
Here we used that $\mathcal{O}(1)$ is ample on $\mathbb{CP}^N$. Why?
$\square$.

• **Prop.** Ample line bundle must be positive.

pf: $kL = i^*\mathcal{O}(1)$. So $c_1(kL) = \frac{1}{2\pi} i^* \omega_{FS}$.
$\Rightarrow c_1(L)$ is represented by $\frac{1}{2\pi k} i^* \omega_{FS} > 0$. $\square$.

Assume that X is cpt cplx mfd.

- Thm (Kodaira) A positive line bundle is ample.

This is a highly non-trivial result, so we won't give the proof here. But we would like to point out the key fact used in the proof.

▲ Let $\mathcal{F}$ be an analytic coherent sheaf on X, L a positive line bundle on X, then one has

Serre Vanishing $H^i(X, \mathcal{F} \otimes L^k) = 0$ for $\forall i > 0$ & $\forall k \gg 0$.

This will imply that

① For $\forall p \in X$, $\exists k \gg 0$ s.t. $\exists s \in H^0(X, kL)$ s.t. $s(p) \neq 0$. Note that this is open condition, So using covering argument, $\exists k \gg 0$, s.t. $\forall p \in X$, $\exists s \in H^0(X, kL)$ s.t. $s(p) \neq 0$. Then $X \to \mathbb{CP}^N$ is well-defined.

So that L is semi-ample.

② For $\forall p, q \in X$, $\exists k \gg 0$ s.t. $\exists s_1, s_2 \in H^0(X, kL)$ s.t. $s_1(p) = 0$, $s_1(q) \neq 0$ while $s_2(p) \neq 0$ & $s_2(q) = 0$.
The covering argument as above shows that, making k larger, the map $X \to \mathbb{CP}^N$ is injective.

③ We need to show that sections in $H^0(X, kL)$ separate tangent directions at $\forall p \in X$, by further increasing k.

For ①, we need to show that $H^0(X, kL) \to kL_p$ is surjective

For ② we need $H^0(X, kL) \to kL_p \oplus kL_q$ surjective

For ③ we need $H^0(X, kL) \to kL_p \otimes \mathcal{O}_p / m_p^2$ surjective

$m_p \subseteq \mathcal{O}_p$ max ideal.

To get surjectivity, it is enough to use

$$H^1(X, kL \otimes m_p) = H^1(X, kL \otimes m_{p,q}) = H^1(X, kL \otimes m_p^2) = 0$$

The exact sequence will finish the proof. $\square$

- To show that $H^i(X, kL \otimes \mathcal{F}_i)$ vanishes for $k \gg 0$, the positivity of L plays key role, which allows us to solve "$\bar\partial$-equations" using Hörmander's L^2 theory.

- <u>Cor</u>. A cpt cplx mfd mfd is projective iff it admits a positive line bundle. (In this case it must be Kähler) iff it admits a Hodge class (i.e., a Kähler class in $H^{1,1}(X,\mathbb{Z})$).

- Finally, we state a few vanishing thm's.

① (Kodaira-Nakano vanishing) If $L \to X$ is positive X cpt. Then $H^q(X, \Omega^p_X \otimes L) = 0$ when $p+q > n$.

② (Kodaira Vanishing) If $L \to X$ is positive. X cpt. then $H^q(X, K_X \otimes L) = 0$ when $q \geq 1$.

weaker than ample

③ (Generalized Kodaira vanishing) If L is big & nef X cpt. then $H^q(X, K_X \otimes L) = 0$ when $q \geq 1$.

$\mathrm{Vol}(L) > 0$ for $\forall \varepsilon > 0$, $\exists \alpha \in c(L)$ s.t. $\alpha \geq -\varepsilon\omega$.

④ (Nadel Vanishing) If L is big, X cpt. Assume that L admits a singular Hermitian metric $h = h_0 e^{-\varphi}$, where h_0 is smooth background metric & φ usc, L^1 s.t.
$$R_{h_0} + i\partial\bar\partial\varphi \geq \varepsilon\omega \text{ in distribution sense.}$$
Define $\mathcal{I}(\varphi) := \{ f \in \mathcal{O} \mid |f|^2 e^{-\varphi} \in L^1_{loc} \}$. multiplier ideal sheaf of φ.

Then $H^q(X, K_X \otimes L \otimes \mathcal{I}(\varphi)) = 0$ when $q \geq 1$.

- <u>Cor</u>. L ample. Then $H^q(X, mL) = 0$ for $q \geq 1$ & $m \gg 1$.

Pf: Observe that $mL - K_X$ is positive, so ample, for $m \gg 1$. So the assertion follows from Kodaira Vanishing. □

- A few words about the proof of Kodaira–Nakano thm.

As we mentioned above, one can prove it using L^2-theory. Another elegant way is to use $\boxed{\text{"Nakano identity"}}$. For a holo. Hermitian l.b. (L, h) on a cpt Kähler mfd (X, ω), one can consider the Chern connection ∇_h on L, s.t. $\nabla_h = \partial_h + \bar\partial$. Then define $\begin{cases} \Delta_{\bar\partial} := \bar\partial\bar\partial^* + \bar\partial^*\bar\partial \\ \Delta_\partial := \partial_h\partial_h^* + \partial_h^*\partial \end{cases}$ They act on L-valued (p,q) forms. Then the Nakano identity says that

$$\Delta_{\bar\partial} = \Delta_\partial + [R_h, \Lambda]$$

$R_h(\cdot) := R_h \wedge \cdot$

$\Lambda := (\omega \wedge \cdot)^*$ adjoint of Lefschetz operator.

Here $[R_h, \Lambda] \in C^\infty(X, \Omega^{p,q}(L)^* \otimes \Omega^{p,q}(L))$ is a tensor that can be explicitly described if one diagonalize R_h at one pt, say $R_h = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$, w.r.t. ω.

Then for $\forall u = \sum_{I,J} u_{IJ}\, dz^I \wedge d\bar z^J \in C^\infty(X, \Omega^{p,q}(L))$, it holds that

$$[R_h, \Lambda]u = \sum_{I,J} \Big(\sum_{i\in I}\lambda_i + \sum_{j\in J}\lambda_j - \sum_{k=1}^n \lambda_k\Big) u_{IJ}\, dz^I \wedge d\bar z^J.$$

Now if L is positive, we may choose $\omega := R_h$ so that

$$[R_h, \Lambda]u = (p+q-n)u.$$

Then we find that, whenever $p+q > n$,

$$\|\bar\partial u\|^2 + \|\bar\partial^* u\|^2 = (\Delta_{\bar\partial} u, u)_{\mathcal{L}} \geq ([R_h, \Lambda]u, u)_{\mathcal{L}} \geq \|u\|^2.$$

So $\forall$ $\bar\partial$-harmonic L-valued (p,q) form must be zero. $\square$

- $\boxed{\text{Asymptotic Riemann-Roch}}$: Let L be ample.

then for $\forall m \gg 1$, one has

$$\dim H^0(X, mL) = \text{Hilbert polynomial of } m \text{ (of deg } n)$$

$$= \frac{L^n}{n!} m^n + \frac{(-K_X)\cdot L^{n-1}}{2(n-1)!} m^{n-1} + \cdots + \chi(\mathcal{O}_X).$$

intersection number

$$\chi(\mathcal{O}_X) := \sum_{i\geq 0} (-1)^i \dim H^i(X, \mathcal{O}_X)$$

pf: By Hirzebruch-Riem-Roch, one has

$$\chi(mL) = \sum_{i\geq 0} (-1)^i \dim H^i(X, mL) = \underline{\int_X \mathrm{ch}(mL)\,\mathrm{td}(X)}$$

Hilbert polynomial

Since $H^i(X, mL) = 0$ for $i \geq 1$ & $m \gg 1$

we find that Hilbert polynomial $= \dim H^0(X, mL)$ for $m \gg 1$. $\square$

- As a consequence, we find that

$$\mathrm{Vol}(L) = L^n \quad \text{when } L \text{ is ample.}$$

so must be noncollapsing

So $\mathrm{Vol}(L) = \int_X \omega^n$ if one pick $\omega \in \frac{1}{2\pi} c_1(L)$. In this case Vol must be integer.

This also holds when L is merely nef ($\dim H^i(X, mL) = o(m^n)$ for $i \geq 1$)

This no longer holds when L is big.