Lecture 9

Kodaira Embedding
Outline.

1. Lefschetz (1,1) theorem.
2. Ample line bundle & positive line bundle.
3. Kodaira vanishing.
4. Kodaira embedding.

• Let $X$ be a complex manifold with a hermitian line bundle $L$ on it. We have seen that, using the inclusion $\mathbb{Z} \to \mathbb{R}$, $c_1(L)$ can be represented by a closed real $(1,1)$-form, a curvature form of $L$.

Now we ask the following question: if $\frac{\pi}{2} \in H^2(X, \mathbb{Z}) \cap \text{Im}(H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{R}))$ can be represented by a closed real $(1,1)$-form $\omega$ that is of type $(1,1)$, then can one find a hermitian line bundle $L$ over $X$ such that $c_1(L) = \frac{\pi}{2}$?

To study this problem, we look at the exact sequence:

$$0 \to \text{Pic}(X) \to H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X) \cong H^0_{\mathbb{Z}}(X).$$

We find that $\text{Ker } \tau = \text{Im } c_1$. So we need to study the map $H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X)$.

In what follows, we will use a larger inclusion $\mathbb{Z} \to \mathbb{R} \to \mathbb{C}$.

\[
\begin{array}{ccc}
\mathbb{R} & \rightarrow & \mathbb{A}^\infty_{\mathbb{R}} \rightarrow \\
\downarrow & & \downarrow \\
\mathbb{C} & \rightarrow & \mathbb{A}^\infty_{\mathbb{C}} \rightarrow \\
\downarrow & & \downarrow \\
\mathcal{O}_X & \rightarrow & \mathbb{A}^\infty_{\mathbb{C}} \rightarrow \\
\end{array}
\]

The diagram is commutative.

From this we get an explicit description of $H^2(X, \mathbb{C}) \to H^2(X, \mathcal{O}_X)$; for a $\mathbb{C}$-valued $d$-closed $2$-form $\omega$, write $\omega = \omega^{0,2} + \omega^{1,1} + \omega^{2,0}$. Then $\int_X \omega^{0,2} = 0$. So $[\omega^{0,2}]$ defines an element in $H^0_{\mathbb{Z}}(X)$.

The map $H^2_{\mathbb{R}}(X, \mathbb{C}) \to H^0_{\mathbb{Z}}(X)$ is well-defined.

Now the map $H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X)$ can be described as follows:
Using Čech cohomology, a \([\alpha] \in H^1(X, \mathbb{Z})\) gives rise to a \(d\)-closed real \(2\)-form on \(X\), so treating it as a \(\mathbb{C}\)-valued \(2\)-form one can take its \((0,2)\) part \(\alpha^{0,2}\).

Then the map \(H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathbb{C})\) is given by \([\alpha] \mapsto [\alpha^{0,2}]\).

This actually holds for \(H^{2k}(X, \mathbb{Z}) \rightarrow H^{2k}(X, \mathbb{C})\), \(k \geq 1\).

Now, what is the kernel of this map?

By the above discussion, \(\forall \xi \in \text{Im}(H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathbb{C}))\) is mapped to zero in \(H^2(X, \mathbb{C})\) iff \(\exists \) (any) representative \(\alpha \in \xi\), its \((1,2)\) part \(\alpha^{1,2}\) is \(\delta\)-exact, hence \([\alpha^{0,2}] = 0\) in \(H^2(X, \mathbb{C})\).

So we see that \(\exists \) hol. \(L \) s.t. \(G(L) = \xi \in \text{Im}(H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathbb{C}))\) iff \(\exists \) (any) representative \(\alpha \in \xi\) satisfies that \(\alpha^{0,2}\) is \(\delta\)-exact.

**Cor.** A class \(\xi \in \text{Im}(H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathbb{C}))\) is the 1\(^{st}\) Chern class of a hol. line bundle iff one can find a representative \(\alpha \in \xi\) s.t. \(\alpha\) is of type \((1,1)\).

**pf.** If \(\xi = G(L)\) for some hol. \(L\), then the Chern curvature form (normalized by \(\frac{1}{2\pi}\)) is of type \((1,1)\), representing \(\xi\).

Conversely, if \(\exists \) \((1,1)\) representative in \(\xi\), we see that \(\xi\) lies in the kernel of \(H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathbb{C})\), so \(\exists \) hol. \(L\) s.t. \(G(L) = \xi\) (modulo torsion of course).

Another way to prove this is to use the fact that \(\forall\) element \(\xi \in \text{Im}(H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathbb{C}))\) corresponds to a cplx line bundle \(L\), so the goal is to find a "holomorphic structure on \(L\", i.e. a \(\tilde{\partial}_L\)-operator" \(\tilde{\partial}_L : A^0(L) \rightarrow A^1(L)\) that satisfies the Leibnitz rules and \(\tilde{\partial}_L^2 = 0\). This is where the type \((1,1)\) condition comes in: first, pick a cplx connection \(\tilde{\nabla}\) on \(L\), then its curvature \(\tilde{\nabla}^2 = \tilde{\omega}\) for some 1-form \(\tilde{\omega}\). Define \(\tilde{\partial}_L\) to be the \((0,1)\) part of \(\tilde{\nabla} + \partial\). Then we check that \(\tilde{\partial}_L^2 = 0\) since \((D + \partial)^2 = 0\) has no type \((0,2)\) part.
In the Kähler setting, the above result is known as Lefschetz theorem on \((1,1)\)-classes, which is stated as follows: let \(X\) be cpt Kähler.

\[
\text{Define } H^{1,1}(X,\mathbb{Z}) := \text{Im} (H^2(X,\mathbb{Z}) \rightarrow H^2(X,\mathbb{C})) \cap H^1_\partial(X)
\]

Then the map \(\text{Pic}(X) \rightarrow H^{1,1}(X,\mathbb{Z})\) is surjective.

\(\text{pf.}\) We go back to the proof of the previous lemma. In the Kähler setting, \(H^0(X,\mathcal{O}) = H^2_\partial(X) \oplus H^1_\partial(X) \oplus H^0_\partial(X)\) for \(\nu \in \text{Im} (H^2(X,\mathbb{Z}) \rightarrow H^2(X,\mathbb{C}))\), we can find a harmonic representative \(\alpha \in \mathcal{H}\) (wrt. some Kähler metric).

Since \(\mathcal{H} \subset H^2(X,\mathbb{C})\), \(\alpha\) is real. Then \(\bar{\alpha} \alpha = \alpha \bar{\alpha}\). Now using the exact sequence

\[
\ldots \rightarrow \text{Pic}(X) \xrightarrow{c_1} H^1(X,\mathbb{Z}) \rightarrow H^1(X,\mathbb{C}) \rightarrow \ldots
\]

\(c_1\) is of the form \(c_1(L)\) iff \(\alpha \bar{\alpha} = 0\) so that \(\alpha = \bar{\alpha}\).

\(\Box\)

- **Def.** A holomorphic line bundle \(L\) over \(X\) is called **positive** if \(\exists\) Hermitian metric \(h\) on \(L\) st. its curvature form \(R_h\) is positive definite (so it gives a Kähler metric).

- **Prop:** A complex manifold admits a positive line bundle iff

\[
\exists \nu \in \text{Im} (H^2(X,\mathbb{Z}) \rightarrow H^2(X,\mathbb{R})) \text{ st. } \exists \text{ Kähler form } \nu.
\]

\(\text{pf.}\) This is clear from the above discussions. \(\Box\)

The following result is also clear:

- **Prop:** A compact Kähler manifold admits a positive line bundle iff \(H^1(X,\mathbb{Z}) \cap \mathcal{K}(X) \neq \emptyset\)

\(\uparrow\) Kähler cone

- **Prop:** Let \(X\) be compact Kähler. If \(h^{0,2} = 0\), then \(X\) admits a positive line bundle.
\[ h^0 = h^1 = 0 \] implies that \( \dim H^2(X, \mathbb{R}) = h^1 \), so \( \text{Im}(H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{R})) \) spans the entire \( H^2(X, \mathbb{R}) \).

Thus there exist line bundles \( L_1, \ldots, L_n \) and \( a_1, \ldots, a_n \in \mathbb{R} \) s.t.
\[ a_1 c_1(L_1) + \cdots + a_n c_1(L_n) = EW. \]

Slightly perturbing \( w \) and \( ai \), we may assume that \( ai \in \mathbb{Q} \).

Then a sufficiently divisible integer \( N \) with \( Na_i \in \mathbb{Z} \)
defines a line bundle \( L = Na_1 L_1 + \cdots + Na_n L_n \)
s.t. \( c_1(L) = [NW] \) is represented by a Kähler form. \( \square \).

**Def.** A line bundle \( L \) over a compact complex manifold is called **ample** if for all \( K > 0 \), a basis \( \{ s_0, \ldots, s_d \} \) of \( H^0(X, \mathcal{O}_L) \) defines an embedding:
\[ X \overset{i}{\hookrightarrow} \mathbb{P}^d, \quad x \mapsto [s_0(x), \ldots, s_d(x)]. \]

For such \( K \), \( kL \) is called **very ample**.

Check that in this case \( kL = i^*(\mathcal{O}(1)) \).

It is clear from the definition that

**Prop.** A compact complex manifold is projective iff it admits an ample line bundle.

**Pf:** One direction is direct from the definition.

For the other direction, if \( X \) embeds in \( \mathbb{P}^N \), then \( \mathcal{O}(1)|_X \) is ample.

Here we used that \( \mathcal{O}(1) \) is ample on \( \mathbb{P}^n \). Why? \( \square \).

**Prop.** Ample line bundles must be positive.

**Pf:** \( kL = i^*(\mathcal{O}(1)) \). So \( c_1(kL) = \frac{1}{2\pi i} \mathcal{O} \cdot \mathcal{O} \).

\[ \Rightarrow c_1(L) \text{ is represented by } \frac{1}{2\pi i} i^* \mathcal{O} \cdot \mathcal{O} > 0. \] \( \square \).
Assume that $X$ is cpt cplx mfd.

Thm (Kodaira) A positive line bundle is ample.

This is a highly non-trivial result, so we won't give the proof here. But we would like to point out the key fact used in the proof.

Let $F$ be an analytic coherent sheaf on $X$, $L$ a positive line bundle on $X$. Then one has

$$H^i(X, F \otimes L^k) = 0 \text{ for } i > 0 \text{ & } k \gg 0.$$ 

This will imply that

1. For $y \in X$, $\exists k \gg 0$ s.t. $\exists s \in H^0(X, yL_k)$ s.t. $s(y) \neq 0$. Note that this is open condition, so using covering argument, $\exists k \gg 0$, $\forall y \in X$, $\exists s \in H^0(X, yL_k)$ s.t. $s(y) \neq 0$. Then $X \to \mathbb{P}^N$ is well-defined.

2. For $y, y' \in X$, $\exists k \gg 0$ s.t. $s_{y} \in H^0(X, yL_k)$ s.t. $s_{y}(p) = 0$, $s_{y'}(q) = 0$ while $s_{y}(p) \neq 0$ & $s_{y'}(q) = 0$. The covering argument as above shows that, making $k$ larger, the map $X \to \mathbb{P}^N$ is injective.

3. We need to show that sections in $H^0(X, kL)$ separate tangent directions at $y \in X$, by further increasing $k$.

For 1, we need to show that $H^0(X, kL) \to kL_y$ is surjective.

For 2 we need $H^0(X, kL) \to kL_y \oplus kL_{y'}$ surjective.

For 3 we need $H^0(X, kL) \to kL_y \oplus O_y/m_y^2$ surjective.

To get surjectivity, it is enough to use

$$H^1(X, kL \otimes m_y) = H^1(X, kL \otimes m_y^2) = 0.$$ 

The exact sequence will finish the proof.
To show that \( H^k(X, K_L \otimes F) \) vanishes for \( k > 0 \), the positivity of \( L \) plays key role, which allows us to solve "\( \bar{\partial} \) -equations" using Hörmander's \( L^2 \) theory.

- Cor. A cpt cpx mtd mfd is projective iff it admits a positive line bundle. (In this case it must be Kähler).
  - iff it admits a Hodge class (i.e., a Kähler class in \( H^k(X, \mathbb{C}) \)).

Finally, we state a few vanishing thms:

1. (Kodaira-Nakano vanishing) If \( L \to X \) is positive \( X \) opt.
   - then \( H^q(X, L^k \otimes L) = 0 \) when \( q + k > n \).

2. (Kodaira Vanishing) If \( L \to X \) is positive \( X \) opt.
   - then \( H^q(X, K_L \otimes L) = 0 \) when \( q \geq 1 \).

3. (Generalized Kodaira vanishing) If \( L \) is big \& nef \( X \) opt.
   - then \( H^q(X, K_L \otimes L) = 0 \) when \( q \geq 1 \).

4. (Nadel Vanishing) If \( L \) is big \( X \) opt.
   - Assume that \( L \) admits a singular Hermitian metric \( h = h_0 e^{-\varphi} \), where \( h_0 \) is smooth background metric \& \( \varphi \) use, \( L \) st.
   - \( \mathbb{R} \alpha + i \delta \alpha \varphi \leq \varepsilon \omega \) in distribution sense.
   - Define \( I(\varphi) := \{ f \in C \mid f \varphi e^{-\varphi} \leq L \text{ on } f \} \).
   - Then \( H^q(X, K_L \otimes L \otimes I(\varphi)) = 0 \) when \( q \geq 1 \).

- Cor. \( L \) ample. Then \( H^q(X, mL) = 0 \) for \( q \geq 1 \) \& \( m \geq 1 \).
  - pf: Observe that \( mL - K_L \) is positive, so ample, for \( m \geq 1 \).
  - So the assertion follows from Kodaira Vanishing.
A few words about the proof of Kodaira-Nakano thm.

As we mentioned above, one can prove it using $L^*$-theory. Another elegant way is to use "Nakano identity". For a holomorphic line bundle $(L,h)$ on a compact Kähler manifold $(X,ω)$, one can consider the Chern connection $∇_h$ on $L$, such that $∇_h = ∇ + ȳ$. Then define $Δ_5 := ∇^* + ∇^* ȳ$. They act on $L$-valued $(p,q)$ forms. Then the Nakano identity says

$$Δ_5 = Δ_2 + [R_u,λ]$$

Here $[R_u,λ] ∈ C^0(X, Ω^{p,q}(T^*X))$ is a tensor that can be explicitly described if one diagonalize $R_u$ at one point, say $R_u = diag(l_1, ..., l_n)$, w.r.t. $ω$.

Then for $\nu = \sum_{i,j} u_{ij} dz^i ∧ d\bar{z}^j ∈ C^0(X, Ω^{p,q}(L))$, it holds that

$$E[R_u,λ] \nu = \sum_{i,j} (\sum_{i,k} l_i + \sum_{j,k} l_j - \sum_{k} l_k) u_{ij} dz^i ∧ d\bar{z}^j.$$

Now if $L$ is positive, we may choose $ω := R_u$ so that

$$E[R_u,λ] \nu = (p+q-n) \nu.$$ 

Then we find that, whenever $p+q ≥ n$,

$$\|\nu\|^2 + \|\bar{\nu}\|^2 = (Δ_5 \nu, ω) ≥ (E[R_u,λ] \nu, ω) ≥ \|\nu\|^2.$$

So all $Δ_5$-harmonic $L$-valued $(p,q)$ form must be zero. □

Asymptotic Riemann-Roch:

Let $L$ be ample, then for $m ≥ 1$, one has

$$\dim H^0(X, mL) = \text{Hilbert polynomial of } m(\text{of degree } n)$$

$$= \frac{l^n}{n!} m^n + \left(\frac{K_X.L_{n-1}}{2(n-1)!}\right) m^{n-1} + \cdots + X(0).$$

$$X(0) := \sum_{i≥0} (-1)^i \dim H^i(X, 0X)$$

Proof: By Hirzebruch-Riemann-Roch, one has

$$X(mL) = \sum_{i≥0} (-1)^i \dim H^i(X, mL) = \int_X ch(L)td(X)$$

Since $H^i(X, mL) = 0$ for $i ≥ 1$ and $m ≥ 1$, we find that Hilbert polynomial $= \dim H^0(X, mL)$ for $m ≥ 1$. □
As a consequence, we find that

$$\text{Vol}(L) = L^n$$ when $L$ is ample.

So $\text{Vol}(L) = \int_X w^n$ if one pick $w \in \mathcal{A}(L)$. In this case $\text{Vol}$ must be integer.

This also holds when $L$ is merely nef ($\dim H^1(X, mL) = o(m^n)$ for all $m$).

This no longer holds when $L$ is big.