Upper Bounds for Tian's Alpha Invariant of Smooth Fano Manifolds

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August 2026

Abstract

Let X be a smooth complex Fano manifold of dimension n, and put $L = -K_X$ and $V = L^n$. I failed to obtain the conjectural inequality $\alpha(X) \leq 1$ in arbitrary dimension. But we record two unconditional and completely explicit estimates:

$$\alpha(X) \leq \min \left\{ 2 \left\lfloor \frac{n}{4} \right\rfloor + 1, \frac{n}{V^{1/n}} \right\}.$$

The first bound follows from Kodaira vanishing, Serre duality, and a root count for the anticanonical Hilbert polynomial; the second follows from asymptotic jet counting at a smooth point. Consequently, $\alpha(X) \leq 2$ whenever $|-2K_X| \neq \varnothing$ or $V \geq (n/2)^n$. In dimensions at most five one has the stronger known bound $\alpha(X) \leq 1$.

We also analyze the first dimension not covered by the known general anticanonical nonvanishing theorem. If a smooth Fano sixfold satisfied $\alpha(X) > 2$, then it would have Fano index one and $1 \leq (-K_X)^6 \leq 6$. Finally, a Nadel–subadjunction argument shows that, in the range $2 < \alpha(X) < 3$, a minimal log canonical center arising from a threshold below three must have dimension at least three. The remaining step is precisely an effective nonvanishing problem on the subadjunction center.

1 Introduction

Let X be a smooth Fano manifold over $\mathbf{C}$, and let $L = -K_X$. Demailly's algebraic characterization of Tian's invariant gives

$$\alpha(X) = \inf_{\substack{m \geq 1 \\ H^0(X, mL) \neq 0}} \inf_{D_m \in |mL|} \operatorname{lct}\left(X; \frac{1}{m} D_m\right); \tag{1}$$

see [4, Appendix A]. Equivalently, if

$$T_L(v) := \sup_{\substack{m \geq 1 \\ 0 \neq s \in H^0(X, mL)}} \frac{v(\operatorname{div}(s))}{m},$$

then

$$\alpha(X) = \inf_v \frac{A_X(v)}{T_L(v)}, \tag{2}$$

where v ranges over divisorial valuations with $T_L(v) > 0$.

The motivating question is whether

$$\alpha(X) \leq 1 \tag{3}$$

for every smooth Fano manifold. It is still listed as Problem 2.5 in the AIM problem list on K-stability [1]. If $|-K_X|$ is nonempty, (3) is immediate by looking at the generic point of a prime component of an anticanonical divisor. Anticanonical nonvanishing is known for smooth Fano manifolds of dimension at most five, but is not known in arbitrary dimension; see [8, 7].

After spending ten hours on this problem, I failed miserably. But let me isolate what can be proved without assuming anticanonical nonvanishing. Our main elementary result is the following.

Theorem 1.1. *Let X be a smooth complex Fano manifold of dimension $n \geq 1$. Set $L = -K_X$ and $V = L^n$. Then*

$$\alpha(X) \leq \min\left\{2\left\lfloor\frac{n}{4}\right\rfloor + 1, \frac{n}{V^{1/n}}\right\}. \tag{4}$$

In particular, each of the following conditions implies $\alpha(X) \leq 2$:

(i) $H^0(X, -2K_X) \neq 0$*;*

(ii) $(-K_X)^n \geq (n/2)^n$.

The first estimate in (4) is a bound for the *bottom weight*

$$b(X) := \min\{m \geq 1 \mid H^0(X, mL) \neq 0\}.$$

It is obtained from the symmetry of the Hilbert polynomial under $t \mapsto -t-1$. The second estimate is the usual point-jet estimate, included with all constants for completeness.

Combining Theorem 1.1 with the known nonvanishing results gives

$$\alpha(X) \leq 1 \quad (n \leq 5), \qquad \alpha(X) \leq 3 \quad (n = 6, 7). \tag{5}$$

The estimate $\alpha \leq 2$ in all dimensions would follow from the special case $H^0(X, -2K_X) \neq 0$ of the Ambro–Kawamata effective nonvanishing conjecture. This implication is only one-way: higher anticanonical systems could in principle prove $\alpha \leq 2$ even when $|-2K_X|$ is empty. We are not aware of an all-dimensional theorem proving $\alpha \leq 2$, nor of a smooth counterexample.

Acknowledgments. This note grew out of my conversation with Kewei Zhang. Honestly speaking, I have no idea why he pushed me so hard on math problems. I am just not strong enough, so I failed him once again. Maybe next month, I will be much stronger to finish this project. The author is partially supported by Kewei Zhang, because for unknown reasons he has generously paid 200$ just to chat with me.

2 The bottom-weight bound

We start with an elementary observation.

Lemma 2.1. *If $H^0(X, mL) \neq 0$ for some $m \geq 1$, then $\alpha(X) \leq m$. More precisely, if $D = \sum_i a_i D_i \in |mL|$, then*

$$\alpha(X) \leq \min_i \frac{m}{a_i}.$$

Proof. At the generic point of D_i, the coefficient of D_i in D/m is a_i/m. Hence

$$\operatorname{lct}\left(X; \frac{1}{m}D\right) \leq \frac{m}{a_i}.$$

Now use (1). $\square$

Proposition 2.2. *For every smooth Fano n-fold,*

$$b(X) \leq 2\left\lfloor \frac{n}{4} \right\rfloor + 1. \tag{6}$$

Consequently,

$$\alpha(X) \leq 2\left\lfloor \frac{n}{4} \right\rfloor + 1.$$

Proof. Consider the Hilbert polynomial

$$P(t) := \chi(X, tL).$$

For every integer $t \geq 0$ we have

$$tL = K_X + (t+1)L,$$

and $(t+1)L$ is ample. Kodaira vanishing therefore gives

$$P(t) = h^0(X, tL) \geq 0 \quad (t \geq 0), \qquad P(0) = 1. \tag{7}$$

On the other hand, Serre duality and $K_X = -L$ yield

$$P(t) = (-1)^n P(-t-1). \tag{8}$$

Put $b = b(X)$ and $r = b - 1$. Then

$$1, 2, \ldots, r$$

are roots of P. By (8), so are

$$-2, -3, \ldots, -r-1.$$

If n is odd, (8) also gives the root $-\frac{1}{2}$. Thus

$$2r + (n \bmod 2) \leq n. \tag{9}$$

For $n = 4q$ or $n = 4q+1$, (9) immediately gives $r \leq 2q$. Suppose next that $n = 4q+2$. Root counting leaves only the additional possibility $r = 2q+1$. In that case all roots of the degree-n polynomial have already been listed, so

$$P(t) = c\prod_{j=1}^{r}(t-j)(t+j+1)$$

with $c > 0$, since the leading coefficient of P is $L^n/n! > 0$. But then $P(0) < 0$ because r is odd, contradicting $P(0) = 1$. Hence $r \leq 2q$. The case $n = 4q+3$ is identical: in the only remaining extremal case $r = 2q+1$, one would have

$$P(t) = c\left(t + \frac{1}{2}\right)\prod_{j=1}^{r}(t-j)(t+j+1), \qquad c > 0,$$

which again gives $P(0) < 0$. We have proved

$$r \leq 2\left\lfloor \frac{n}{4} \right\rfloor,$$

and hence (6). The alpha bound follows from Lemma 2.1. □

Remark 2.3. The Hilbert-polynomial argument cannot by itself force $H^0(X, 2L) \neq 0$ in dimension six. Indeed,

$$P(t) = \frac{(t-1)(t-2)(t+2)(t+3)(t^2+t+60)}{720} \tag{10}$$

satisfies $P(0) = 1$, $P(1) = P(2) = 0$, $P(t) = P(-t-1)$, has positive leading coefficient $1/6!$, and is integer-valued and positive for every integer $t \geq 3$. For example, its Newton expansion is

$$P(t) = 1 - \binom{t}{1} + \binom{t}{2} + 5\binom{t}{3} + 5\binom{t}{4} + 3\binom{t}{5} + \binom{t}{6}.$$

This is only a formal numerical model, not the Hilbert polynomial of a known Fano sixfold. It nevertheless shows that Kodaira vanishing, Serre duality, and integrality of the Hilbert polynomial do not suffice to lower the bound 3 to 2.

3 The volume–jet bound

Proposition 3.1. *Let X be a smooth Fano n-fold and $V = (-K_X)^n$. Then*

$$\alpha(X) \leq \frac{n}{V^{1/n}}. \tag{11}$$

Proof. Fix a point $x \in X$ and a real number $0 < \tau < V^{1/n}$. Set $k_m = \lfloor \tau m \rfloor$. Asymptotic Riemann–Roch and the local Hilbert–Samuel formula give

$$h^0(X, mL) = \frac{V}{n!} m^n + O(m^{n-1}),$$
$$\ell\left(\mathcal{O}_{X,x}/\mathfrak{m}_x^{k_m}\right) = \frac{\tau^n}{n!} m^n + O(m^{n-1}).$$

Since $\tau^n < V$, the jet-evaluation map

$$H^0(X, mL) \longrightarrow \mathcal{O}_{X,x}/\mathfrak{m}_x^{k_m}$$

has a nonzero kernel for all sufficiently large m. We may therefore choose $D_m \in |mL|$ with

$$\operatorname{mult}_x D_m \geq k_m.$$

Let E be the exceptional divisor of the blow-up of X at x. Since x is smooth, $A_X(E) = n$, and hence

$$\operatorname{lct}_x(X; D_m) \leq \frac{n}{\operatorname{mult}_x D_m} \leq \frac{n}{k_m}.$$

After rescaling the divisor,

$$\alpha(X) \leq \operatorname{lct}\left(X; \frac{1}{m} D_m\right) = m \operatorname{lct}(X; D_m) \leq \frac{nm}{k_m}.$$

Letting $m \to \infty$ gives $\alpha(X) \leq n/\tau$, and then $\tau \nearrow V^{1/n}$ proves (11). □

Corollary 3.2. *One has $\alpha(X) \leq 2$ if either*

$$H^0(X, -2K_X) \neq 0$$

or

$$(-K_X)^n \geq \left(\frac{n}{2}\right)^n.$$

Proof. The second assertion follows from Proposition 3.1. For the first, apply Lemma 2.1 with $m = 2$. □

Remark 3.3. If $\alpha(X) > 2$, Proposition 3.1 forces the strict volume inequality

$$(-K_X)^n < \left(\frac{n}{2}\right)^n.$$

Thus the point-jet method fails exactly in the small-volume regime.

4 A Nadel–subadjunction reduction

The next statement explains why a natural attempt to prove $\alpha \leq 2$ becomes another instance of effective nonvanishing.

Proposition 4.1. *Assume $H^0(X, 2L) = 0$. Let $B \sim_{\mathbf{Q}} cL$, where $2 < c < 3$, and suppose that (X, B) is log canonical with an exceptional minimal log canonical center Z. Then there is an effective $\mathbf{Q}$-divisor Δ_Z such that (Z, Δ_Z) is klt and*

$$K_Z + \Delta_Z \sim_{\mathbf{Q}} (c-1)L|_Z, \tag{12}$$

while

$$H^0(Z, 2L|_Z) = 0. \tag{13}$$

Consequently, the data $(Z, \Delta_Z; 2L|_Z)$ violate the predicted conclusion of Ambro–Kawamata effective nonvanishing.

Proof. Since $K_X = -L$, we have

$$2L - (K_X + B) \sim_{\mathbf{Q}} (3-c)L, \tag{14}$$

which is ample. The standard extension theorem for an exceptional log canonical center, obtained from Nadel vanishing after tie-breaking, therefore gives a surjection

$$H^0(X, 2L) \longrightarrow H^0(Z, 2L|_Z).$$

The source is zero, proving (13). Kawamata subadjunction for exceptional minimal log canonical centers [11, 6] gives an effective boundary Δ_Z satisfying (12) and such that (Z, Δ_Z) is klt. Restricting (14) to Z gives

$$2L|_Z - (K_Z + \Delta_Z) \sim_{\mathbf{Q}} (3-c)L|_Z,$$

which is ample. Thus all the positivity hypotheses of effective nonvanishing hold on Z, but (13) says that its conclusion fails. □

Corollary 4.2. *Assume $2 < \alpha(X) < 3$. Choose a threshold boundary whose coefficient lies in $(2, 3)$, and make a chosen minimal log canonical center exceptional by the standard tie-breaking perturbation. Then that center has dimension at least 3.*

Proof. Choose an effective $D \sim_{\mathbf{Q}} L$ such that

$$2 < \operatorname{lct}(X; D) < 3.$$

Such a divisor exists by the definition of $\alpha(X)$. At the threshold, tie-breaking produces a boundary as in Proposition 4.1, with its numerical coefficient still in $(2, 3)$. Effective nonvanishing is known for klt pairs in dimensions at most two [10]. Proposition 4.1 therefore excludes $\dim Z \leq 2$. □

Remark 4.3. Subadjunction does not reduce the problem to a lower-dimensional Fano variety: equation (12) shows that $K_Z + \Delta_Z$ is ample. The center is of log general type, and the missing input is effective nonvanishing for the adjoint divisor $2L|_Z$. Moreover, when $\alpha(X) = 3$, the class in (14) becomes numerically trivial. Nadel vanishing and the extension argument lose their strict positivity exactly at this endpoint.

5 The six-dimensional obstruction

We now record additional restrictions in the first dimension not covered by the known general anticanonical nonvanishing result.

Proposition 5.1. *Let X be a smooth Fano sixfold. If $\alpha(X) > 2$, then X has Fano index one and*

$$1 \le V := (-K_X)^6 \le 6. \tag{15}$$

More precisely, with $L = -K_X$ and $x = t(t+1)$, its anticanonical Hilbert polynomial would necessarily be

$$P(t) = \chi(X, tL) = \frac{(x-2)(x-6)(Vx+60)}{720}, \tag{16}$$

and its Chern numbers would satisfy

$$c_2(X)L^4 = 24 - 3V, \tag{17}$$

$$3L^2c_2(X)^2 + L^3c_3(X) - L^2c_4(X) = 21V - 936. \tag{18}$$

Proof. The assumption $\alpha(X) > 2$ and Lemma 2.1 imply

$$P(1) = P(2) = 0.$$

Since $n = 6$, Serre duality gives $P(t) = P(-t-1)$, so P is a cubic polynomial in $x = t(t+1)$. The two vanishings give the factors $(x-2)(x-6)$. The leading coefficient of $P(t)$ is $V/6!$, while $P(0) = 1$. These two conditions determine the remaining linear factor and give (16).

Hirzebruch–Riemann–Roch gives

$$P(t) = \int_X e^{tL}\,\mathrm{td}(X).$$

The coefficient of t^4 is

$$\frac{V + c_2(X)L^4}{288}.$$

On the other hand, expanding (16) gives the coefficient $(60-5V)/720$. Equating the two proves (17). Comparing the coefficient of t^2, using

$$\mathrm{td}_4(X) = \frac{-c_1^4 + 4c_1^2c_2 + 3c_2^2 + c_1c_3 - c_4}{720},$$

then gives (18).

Since $\alpha(X) > 2 > 6/7$, Tian's criterion [13] implies that X admits a positive Kähler–Einstein metric. The Kähler–Einstein Chern–Weil inequality is

$$(2(n+1)c_2(X) - nc_1(X)^2)c_1(X)^{n-2} \ge 0. \tag{19}$$

See, for example, [5, Appendix C, formula (C.1)]. For $n = 6$, substituting $c_1(X) = L$ and (17) gives

$$14c_2(X)L^4 - 6V = 336 - 48V \ge 0.$$

Thus $V \leq 7$. If $V = 7$, equality holds in (19). The equality case has constant positive holomorphic sectional curvature, so the universal cover of X is $\mathbf{P}^6$. A smooth Fano manifold is simply connected; hence $X \simeq \mathbf{P}^6$. This is impossible because

$$(-K_{\mathbf{P}^6})^6 = 7^6 \neq 7.$$

Therefore $V \leq 6$. Since V is a positive integer, this proves (15).

Finally, if the Fano index were $r \geq 2$, we could write $L = rH$ for an ample Cartier divisor H, and then

$$V = r^6 H^6 \geq 2^6,$$

contradicting $V \leq 6$. Thus the Fano index is one. □

Remark 5.2. The restrictions in Proposition 5.1 do not yet give a contradiction for $V = 1, \ldots, 6$. For example, setting $V = 4$ in (16) produces an integer-valued Hilbert polynomial with nonnegative values for $t \geq 3$, and (17) gives $c_2 L^4 = 12$, which is compatible with the strict Chern–Weil inequality. Twisted indices $\chi(\Omega_X^p \otimes mL)$ introduce higher Chern numbers for which Kodaira–Akizuki–Nakano vanishing gives no useful sign. A new curvature inequality controlling the combination in (18), or an independent effective nonvanishing theorem, would be needed to finish this route.

6 Status of the constant two problem

The implication

$$H^0(X, -2K_X) \neq 0 \quad \Longrightarrow \quad \alpha(X) \leq 2$$

is elementary, but the premise is not known for arbitrary-dimensional smooth Fano manifolds. It is a special case of Ambro–Kawamata effective nonvanishing: for $D = -2K_X$, the divisor

$$D - K_X = -3K_X$$

is ample. Recent work continues to treat effective nonvanishing as a conjecture and verifies it for special classes; see, for example, [9, 12].

It is important not to infer that $\alpha(X) \leq 2$ is equivalent to $|-2K_X| \neq \varnothing$. A sufficiently singular member of $|-mK_X|$ for $m \geq 3$ could also prove the desired bound. Conversely, the standard bounded-complement theorem only supplies a complement of an index depending on the dimension, and its log canonical pair may still be klt. This is exactly the exceptional regime in Birkar's terminology [2].

Our literature search, completed in August 2026, found no smooth example with ordinary alpha invariant greater than two. Examples with large equivariant alpha invariant α_G do not contradict this statement, since α_G tests only G-invariant divisors and may be much larger than the ordinary alpha invariant. Singular exceptional Fano varieties can have very large ordinary alpha invariant [14]; these varieties are singular and hence do not answer the smooth question. Lower semicontinuity of the alpha invariant in $\mathbf{Q}$-Gorenstein families [3] does, however, show that a $\mathbf{Q}$-Gorenstein smoothing of a singular fiber with alpha invariant greater than two would produce nearby smooth counterexamples.

Thus the rigorous all-dimensional conclusion of this note is Theorem 1.1. Proving the uniform constant two would require a genuinely new input in the remaining high-dimensional, small-volume regime.

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