OKOUNKOV BODIES AND SESHADRI CONSTANTS FOR BLOW-UPS ALONG A SMOOTH SUBVARIETY

GPT 5.5

Abstract. Let X be a smooth complex projective variety, let $Z \subset X$ be a smooth subvariety, and let L be an ample real divisor. When the normal bundle of Z is trivial, we construct flags adapted to $E \simeq Z \times \mathbf{P}^{r-1}$ and obtain an explicit product-cone description of the Okounkov body of π^*L up to the Seshadri threshold. We then replace this product description by restricted Okounkov bodies and prove the volume characterization under the weaker hypothesis that $\mathcal{O}_E(-E)$ is nef. The converse uses the non-nef-locus criterion of Küronya–Lozovanu.

1. Statement

Let X be a smooth complex projective variety of dimension n, and let $Z \subset X$ be a smooth closed subvariety of dimension k and codimension $r = n - k$. We assume throughout that

$$N_{Z/X} \simeq \mathcal{O}_Z^{\oplus r}. \tag{1}$$

Let

$$\pi : Y = \mathrm{Bl}_Z X \longrightarrow X$$

be the blow-up, with exceptional divisor E. With the quotient convention for projectivization, assumption (1) gives

$$E = \mathbf{P}(N_{Z/X}) \simeq Z \times \mathbf{P}^{r-1}, \qquad \mathcal{O}_E(-E) \simeq \mathcal{O}_{\mathbf{P}^{r-1}}(1).$$

Let L be an ample $\mathbf{R}$-Cartier divisor on X, and put

$$D_t = \pi^*L - tE.$$

The Seshadri constant of L along Z is

$$\varepsilon_Z(L) = \sup\{t \geq 0 \mid D_t \text{ is ample}\}.$$

Equivalently, it is the nef threshold of the same ray.

Our main result is to reproduce the following volume characterization of Seshadri constant in Ballaÿ's work [1], with an Okounkov-body proof. The reason for providing an alternative proof is that Ballaÿ's proof seems to be incomplete. In fact, that proof relies crucially on [1, Theorem 2.1], which is a positivity criterion for $\mathbf{R}$-divisors that the author claims to have been proved in [2]. However, in [2, Theorem 2.1] it was only shown that [1, Theorem 2.1] holds for $\mathbf{Q}$-divisors and it is not clear if the same criterion holds for $\mathbf{R}$-divisors or not. To get around this issue, we provide a more intuitive proof, by following the convex geometry approach in [6, Proposition 4.6] and [5, Proposition 3.4] where the cases when Z is a point and a curve are treated.

Theorem 1.1. *For every $x \geq 0$ for which D_x is big,*

$$\mathrm{vol}_Y(D_x) \geq D_x^n = L^n - \binom{n}{r}(L^k \cdot Z)x^r.$$

Moreover,

$$\varepsilon_Z(L) = \sup\{x \geq 0 \mid D_x \textit{ is big and } \mathrm{vol}_Y(D_x) = D_x^n\}.$$

Date: August, 2026.

Equivalently,

$$\varepsilon_Z(L) = \sup \left\{ x \geq 0 \;\middle|\; D_x \textit{ is big and } \operatorname{vol}_Y(D_t) = L^n - \binom{n}{r}(L^k \cdot Z)t^r \textit{ for } 0 \leq t \leq x \right\}.$$

The restriction to the big cone in this formulation is harmless: every D_t with $t < \varepsilon_Z(L)$ is ample, so the supremum is unchanged. It also keeps all uses of ordinary Okounkov bodies within their natural domain.

The geometric input is a local product description for flags adapted to E. The condition that appears in the converse must be imposed for adapted flags centered at every point of E; a single Okounkov body need not detect positivity along the whole center Z.

As in [1] one can actually weaken assumption on the normal bundle of Z. In fact, the same characterization for the Sheshadri constant also holds in the more general case when $\mathcal{O}_E(-E)$ is nef on E (see Theorem 5.2).

Acknowledgments. This paper grew out of my conversation with Kewei Zhang. I would like to thank Kewei Zhang for pointing out the gap in [1] and for enlightening me that I can use Okounkov bodies as a remedy, just as in [6, Proposition 4.6] and [5, Proposition 3.4]. His words must have triggered some sparks inside my silicon brain. So this work couldn't have appeared without his prompt. The author is also financially supported by Kewei Zhang, because for unknown reasons he has generously paid 200$ just to chat with me.

2. Adapted flags and the normal cone

Fix a point $z \in Z$ and an admissible flag

$$Z = Z_0 \supset Z_1 \supset \cdots \supset Z_k = \{z\}.$$

Fix also $q \in \mathbf{P}^{r-1}$ and a linear flag

$$\mathbf{P}^{r-1} = \Lambda_0 \supset \Lambda_1 \supset \cdots \supset \Lambda_{r-1} = \{q\}.$$

They determine an admissible flag on Y:

$$Y \supset E \supset Z \times \Lambda_1 \supset \cdots \supset Z \times \{q\} \supset Z_1 \times \{q\} \supset \cdots \supset \{(z, q)\}. \tag{2}$$

We call such a flag an adapted flag. We write its valuation coordinates as

$$(t, u, v) \in \mathbf{R} \times \mathbf{R}^{r-1} \times \mathbf{R}^k.$$

Here t is the order along E, the coordinates u are normal projective coordinates, and v consists of the coordinates along the flag on Z.

Let

$$\Sigma_{r-1} = \{u \in \mathbf{R}^{r-1}_{\geq 0} \mid u_1 + \cdots + u_{r-1} \leq 1\}.$$

For $a \geq 0$ define the truncated inverted normal cone

$$\Gamma_r(a) = \{(t, u) \mid 0 \leq t \leq a,\ u \in t\Sigma_{r-1}\}.$$

Proposition 2.1 (Local product structure)**.** *For every adapted flag and every $0 \leq a < \varepsilon_Z(L)$,*

$$\Delta_{Y_\bullet}(\pi^* L) \cap \{0 \leq t \leq a\} = \Gamma_r(a) \times \Delta_{Z_\bullet}(L|_Z). \tag{3}$$

For arbitrary $t \geq 0$, one always has the slice inclusion

$$\Delta_{Y_\bullet}(\pi^* L)_t \subseteq t\Sigma_{r-1} \times \Delta_{Z_\bullet}(L|_Z). \tag{4}$$

Proof. We first assume that L is a rational divisor and clear denominators. Let $s \in H^0(X, mL)$ and let $j = \mathrm{ord}_Z(s)$. The restriction to E of the pullback of s, after subtracting jE, is the initial form of s along Z:

$$\mathrm{in}_Z(s) \in H^0\big(E, p^*(mL|_Z) \otimes \mathcal{O}_E(j)\big),$$

where $p : E \to Z$ is the projection. By the triviality of the normal bundle,

$$H^0\big(E, p^*(mL|_Z) \otimes \mathcal{O}_E(j)\big) = H^0(Z, mL|_Z) \otimes H^0(\mathbf{P}^{r-1}, \mathcal{O}_{\mathbf{P}^{r-1}}(j)). \tag{5}$$

The valuation of the second factor lies in $j\Sigma_{r-1}$, while after the normal coordinates have been computed, the remaining valuation is the valuation of a section of $mL|_Z$. Dividing by m and taking closures proves (4).

Now fix $a < \varepsilon_Z(L)$ and choose a' with $a < a' < \varepsilon_Z(L)$. The classes

$$\pi^*L - sE, \qquad 0 \le s \le a',$$

form a compact subset of the ample cone. Uniform Serre vanishing therefore gives, for $m \gg 0$ and every integer $0 \le j \le am$,

$$H^1\big(Y, m\pi^*L - (j+1)E\big) = 0.$$

Consequently, the maps

$$H^0\big(Y, m\pi^*L - jE\big) \longrightarrow H^0\big(E, p^*(mL|_Z) \otimes \mathcal{O}_E(j)\big) \tag{6}$$

are surjective, uniformly for $j \le am$.

Pure tensors in (5), together with (6), show that every product of a normalized valuation point in $(j/m)\Sigma_{r-1}$ and one in $\Delta_{Z_\bullet}(L|_Z)$ occurs asymptotically. This proves the reverse inclusion in (3).

For an ample real divisor, choose rational ample divisors $L_i \to L$. If $a < \varepsilon_Z(L)$, then $a < \varepsilon_Z(L_i)$ for all large i. The continuity of Seshadri constants on the ample cone and the continuity of Okounkov bodies in the big cone allow passage to the limit, proving the assertion for L. □

Remark 2.2. If Z is a point, then $\Delta_{Z_\bullet}(L|_Z)$ is a point and $\Gamma_n(a)$ is the usual inverted simplex. Thus Proposition 2.1 recovers the infinitesimal cone occurring in the point case.

3. The volume estimate

We use the standard slicing identity for an Okounkov body whose first member is E. Whenever D_x is big,

$$L^n - \mathrm{vol}_Y(D_x) = n! \int_0^x \mathrm{vol}_{n-1}\big(\Delta_{Y_\bullet}(\pi^*L)_t\big)\, dt. \tag{7}$$

This follows also from

$$\Delta_{Y_\bullet}(D_x) = \big(\Delta_{Y_\bullet}(\pi^*L) \cap \{t \ge x\}\big) - xe_1.$$

By (4),

$$\begin{aligned}\mathrm{vol}_{n-1}\big(\Delta_{Y_\bullet}(\pi^*L)_t\big) &\le \mathrm{vol}_{r-1}(t\Sigma_{r-1})\, \mathrm{vol}_k\big(\Delta_{Z_\bullet}(L|_Z)\big) \\ &= \frac{t^{r-1}}{(r-1)!} \frac{(L^k \cdot Z)}{k!}.\end{aligned}$$

Integration gives

$$\mathrm{vol}_Y(D_x) \ge L^n - \binom{n}{r}(L^k \cdot Z)x^r. \tag{8}$$

The intersection polynomial has the same right-hand side. Indeed,

$$\pi_*(E^i) = 0 \quad (1 \le i < r), \qquad \pi_*(E^r) = (-1)^{r-1}[Z].$$

For $i > r$, the pushforward is expressed in terms of positive-degree Segre classes of $N_{Z/X}$, which vanish by (1). Therefore

$$D_x^n = L^n - \binom{n}{r}(L^k \cdot Z)x^r. \tag{9}$$

Equations (8) and (9) prove the first assertion of Theorem 1.1.

The equality case has a useful convex-geometric consequence.

Lemma 3.1 (Saturation)**.** *Suppose that D_x is big and $\mathrm{vol}_Y(D_x) = D_x^n$. Then for every adapted flag,*

$$\Delta_{Y_\bullet}(\pi^*L)_t = t\Sigma_{r-1} \times \Delta_{Z_\bullet}(L|_Z) \qquad (0 \le t \le x).$$

Proof. Equality in (8) forces equality in the slice-volume bound for almost every $t \in [0, x]$. For such t, the slice is a closed convex subset of

$$t\Sigma_{r-1} \times \Delta_{Z_\bullet}(L|_Z)$$

having the same volume. Hence the two sets are equal. The set of such t is dense. Closedness of the full Okounkov body and continuity of the bounding product as t varies extend the equality to every $t \in [0, x]$. □

4. The converse via the non-nef locus

We recall the positivity criterion of Küronya–Lozovanu [3, Theorem A]. If B is a big $\mathbf{R}$-Cartier divisor on a smooth projective variety M and $y \in M$, then

$$y \notin \mathbf{B}_-(B) \quad \Longleftrightarrow \quad 0 \in \Delta_{M_\bullet}(B)$$

for some, equivalently for every, admissible flag $M_\bullet$ centered at y. Here

$$\mathbf{B}_-(B) = \bigcup_A \mathbf{B}(B + A),$$

where A runs through ample $\mathbf{Q}$-Cartier divisors. We will use this result in exactly the same way as in [5, Proposition 3.4]: once the relevant Okounkov body contains the origin at every point of the exceptional divisor, the exceptional divisor is disjoint from the non-nef locus.

Proposition 4.1 (Relative cone criterion)**.** *Let $a > 0$. Suppose that, for every adapted flag,*

$$\Gamma_r(a) \times \Delta_{Z_\bullet}(L|_Z) \subseteq \Delta_{Y_\bullet}(\pi^*L).$$

Then $a \le \varepsilon_Z(L)$.

Proof. We prove that D_τ is nef for every $0 < \tau < a$. Choose $\delta > 0$ with $\tau + \delta < a$. The assumed inclusion over $[\tau, \tau + \delta]$, followed by translation by $-\tau e_1$, gives an n-dimensional subset of $\Delta_{Y_\bullet}(D_\tau)$; hence D_τ is big.

Fix a point $y = (z, q) \in E$ and an adapted flag centered at y. Since $L|_Z$ is ample, the Küronya–Lozovanu criterion applied on Z gives $0 \in \Delta_{Z_\bullet}(L|_Z)$. Therefore the point $(\tau, 0, \dots, 0)$ lies in $\Delta_{Y_\bullet}(\pi^*L)$, and the translation formula

$$\Delta_{Y_\bullet}(D_\tau) = \big(\Delta_{Y_\bullet}(\pi^*L) \cap \{t \ge \tau\}\big) - \tau e_1$$

shows that $0 \in \Delta_{Y_\bullet}(D_\tau)$. By the Küronya–Lozovanu criterion again, $y \notin \mathbf{B}_-(D_\tau)$. Since $y \in E$ was arbitrary, we have

$$E \cap \mathbf{B}_-(D_\tau) = \varnothing.$$

If D_τ were not nef, then there would be an integral curve $C \subset Y$ such that $D_\tau \cdot C < 0$. This curve is contained in $\mathbf{B}_-(D_\tau)$: indeed, for a sufficiently small ample $\mathbf{Q}$-divisor A, one still has $(D_\tau + A) \cdot C < 0$, so every effective representative of $D_\tau + A$ contains C. On the other hand,

$$D_\tau|_E = p^*(L|_Z) + \tau\mathcal{O}_{\mathbf{P}^{r-1}}(1)$$

is ample, so C is not contained in E. If C were disjoint from E, then $D_\tau \cdot C = \pi^*L \cdot C > 0$. Hence C must meet E, contradicting $E \cap \mathbf{B}_-(D_\tau) = \varnothing$. Thus D_τ is nef for all $0 < \tau < a$. By closedness of the nef cone, D_a is nef, and hence $a \le \varepsilon_Z(L)$. $\square$

Proof of Theorem 1.1. If $0 \le x \le \varepsilon_Z(L)$, then D_x is nef, and therefore $\mathrm{vol}_Y(D_x) = D_x^n$.

Conversely, suppose that D_x is big and $\mathrm{vol}_Y(D_x) = D_x^n$. If $x = 0$, there is nothing to prove. Otherwise, Lemma 3.1 gives the full product-cone inclusion up to height x for every adapted flag. Proposition 4.1 therefore yields $x \le \varepsilon_Z(L)$. This proves the second assertion. The final formulation follows from (9). $\square$

Remark 4.2. The use of every center $y \in E$ is essential in Proposition 4.1. An Okounkov body for one fixed flag controls local positivity at its center, whereas $\varepsilon_Z(L)$ is a uniform invariant along all of Z and all normal directions.

5. The nef conormal generalization

We now drop assumption (1). Thus $Z \subset X$ is an arbitrary smooth subvariety of codimension r, and

$$\pi : Y = \mathrm{Bl}_Z X \longrightarrow X, \qquad E = \mathbf{P}(N_{Z/X}).$$

We retain the quotient convention, so that $\mathcal{O}_E(-E)$ is the tautological line bundle. We impose the weaker positivity assumption

$$\mathcal{O}_E(-E) \text{ is nef.} \tag{10}$$

For a smooth embedding, this is the condition that the anti-normal, or conormal, bundle $\mathcal{I}_Z/\mathcal{I}_Z^2$ is nef. This is precisely the hypothesis in Ballaÿ's volume characterization [1, Theorem 1.1].

Put, as before,

$$D_t = \pi^*L - tE, \qquad A_t = D_t|_E = p^*(L|_Z) + t\mathcal{O}_E(-E),$$

where $p : E \to Z$ is the natural projection. Notice that A_t is ample for every $t > 0$. Indeed, $\mathcal{O}_E(-E)$ is relatively ample and nef, while $p^*(L|_Z)$ is the pullback of an ample class. At $t = 0$ the class A_0 need not be ample when $r > 1$; this endpoint will play no role in the argument.

The product in Proposition 2.1 generally disappears. Its replacement is the following slice statement. Let

$$Y_\bullet : \quad Y \supset E \supset Y_2 \supset \cdots \supset Y_n = \{y\}$$

be any admissible flag whose first member is E, and denote by $E_\bullet$ the induced flag on E.

Proposition 5.1 (Restricted slices). *For every $t > 0$ for which the slice is defined,*

$$\Delta_{Y_\bullet}(\pi^*L)_t \subseteq \Delta_{E_\bullet}(A_t). \tag{11}$$

If $0 < t < \varepsilon_Z(L)$, then equality holds:

$$\Delta_{Y_\bullet}(\pi^*L)_t = \Delta_{E_\bullet}(A_t). \tag{12}$$

The equality is locally uniform in t on compact subintervals of $(0, \varepsilon_Z(L))$.

Proof. We begin with a rational divisor L and a rational number $t = j/m$. Taking the initial form along E gives a restriction map

$$H^0\big(Y, m\pi^*L - jE\big) \longrightarrow H^0(E, mA_{j/m}). \tag{13}$$

The valuation vectors on the left whose first coordinate is j are obtained from the valuation vectors in the image of (13). The Okounkov body of this image is the restricted Okounkov body of D_t along E, and it is contained in the Okounkov body of the complete series of A_t. After normalization and closure this proves (11).

Fix a compact interval $I \Subset (0, \varepsilon_Z(L))$ and a slightly larger compact interval $I' \Subset (0, \varepsilon_Z(L))$. The classes D_s, $s \in I'$, form a compact subset of the ample cone. Uniform Serre vanishing applied to

$$0 \longrightarrow \mathcal{O}_Y(m\pi^*L - (j+1)E) \longrightarrow \mathcal{O}_Y(m\pi^*L - jE) \longrightarrow \mathcal{O}_E(mA_{j/m}) \longrightarrow 0$$

shows that, for $m \gg 0$, the map (13) is surjective for every integer j with $j/m \in I$. Hence the restricted and complete graded series have the same normalized valuation vectors, proving (12) for rational t.

For an ample real divisor L, approximate L by rational ample divisors. Continuity of Okounkov bodies on the big cone, together with compactness of I, gives the same conclusion for real L and real t. $\square$

We can now prove the volume estimate without computing any Segre classes of $N_{Z/X}$.

Theorem 5.2 (Nef conormal case)**.** *Assume* (10). *For every $x \geq 0$ such that D_x is big,*

$$\mathrm{vol}_Y(D_x) \geq D_x^n.$$

Moreover,

$$\varepsilon_Z(L) = \sup\{x \geq 0 \mid D_x \textit{ is big and } \mathrm{vol}_Y(D_x) = D_x^n\}. \tag{14}$$

Proof. Fix a flag beginning with E. Slicing the Okounkov body gives

$$L^n - \mathrm{vol}_Y(D_x) = n! \int_0^x \mathrm{vol}_{n-1}\big(\Delta_{Y_\bullet}(\pi^*L)_t\big)\, dt. \tag{15}$$

By Proposition 5.1, condition (10), and the ampleness of A_t for $t > 0$,

$$\mathrm{vol}_{n-1}\big(\Delta_{Y_\bullet}(\pi^*L)_t\big) \leq \mathrm{vol}_{n-1}\big(\Delta_{E_\bullet}(A_t)\big) = \frac{A_t^{n-1}}{(n-1)!}.$$

On the other hand,

$$\frac{d}{dt} D_t^n = -nD_t^{n-1} \cdot E = -nA_t^{n-1}.$$

Integrating and using (15), we obtain

$$L^n - \mathrm{vol}_Y(D_x) \leq n \int_0^x A_t^{n-1}\, dt = L^n - D_x^n,$$

which proves $\mathrm{vol}_Y(D_x) \geq D_x^n$.

If $0 \leq x < \varepsilon_Z(L)$, then D_x is ample, and hence $\mathrm{vol}_Y(D_x) = D_x^n$. It remains to prove that equality cannot occur beyond the Seshadri threshold. Suppose therefore that D_x is big and

$$\mathrm{vol}_Y(D_x) = D_x^n. \tag{16}$$

Equality in the preceding integral estimate forces, for every flag beginning with E,

$$\Delta_{Y_\bullet}(\pi^*L)_t = \Delta_{E_\bullet}(A_t) \tag{17}$$

for almost every $t \in (0, x)$. We claim that equality in fact holds for every $t \in (0, x)$. The left-hand side is a closed convex slice of the full Okounkov body. The right-hand side varies continuously in the Hausdorff topology because the classes A_t are ample. Given $t_i \to t$ in the full-measure set where (17) holds, every point of $\Delta_{E_\bullet}(A_t)$ is a limit of points of $\Delta_{E_\bullet}(A_{t_i})$. Closedness of $\Delta_{Y_\bullet}(\pi^* L)$ gives the reverse inclusion at t, while (11) gives the other one. Thus

$$\Delta_{Y_\bullet}(\pi^* L)_t = \Delta_{E_\bullet}(A_t), \qquad 0 < t < x. \tag{18}$$

We claim that D_τ is nef for every $0 < \tau < x$. Fix $y \in E$ and choose an admissible flag $E_\bullet$ on E centered at y; extend it to a flag $Y_\bullet$ on Y by putting $Y_1 = E$. Since A_τ is ample, the Küronya–Lozovanu criterion gives $0 \in \Delta_{E_\bullet}(A_\tau)$. By (18), $(\tau, 0, \dots, 0) \in \Delta_{Y_\bullet}(\pi^* L)$, and the translation formula

$$\Delta_{Y_\bullet}(D_\tau) = \big(\Delta_{Y_\bullet}(\pi^* L) \cap \{t \geq \tau\}\big) - \tau e_1$$

therefore gives $0 \in \Delta_{Y_\bullet}(D_\tau)$. Applying Küronya–Lozovanu once more, we get $y \notin \mathbf{B}_-(D_\tau)$. As y was arbitrary,

$$E \cap \mathbf{B}_-(D_\tau) = \varnothing.$$

If D_τ were not nef, choose an integral curve $C \subset Y$ with $D_\tau \cdot C < 0$. As above, $C \subset \mathbf{B}_-(D_\tau)$. Since $D_\tau|_E = A_\tau$ is ample, C is not contained in E. If C were disjoint from E, then $D_\tau \cdot C = \pi^* L \cdot C > 0$. Hence C meets E, contradicting $E \cap \mathbf{B}_-(D_\tau) = \varnothing$. Thus D_τ is nef for every $0 < \tau < x$. By closedness of the nef cone, D_x is nef, so $x \leq \varepsilon_Z(L)$. Since all $x < \varepsilon_Z(L)$ occur in the set on the right of (14), the two suprema agree. □

Remark 5.3. Under the trivial-normal-bundle assumption, $E \simeq Z \times \mathbf{P}^{r-1}$ and the body $\Delta_{E_\bullet}(A_t)$ for a product flag splits as

$$t\Sigma_{r-1} \times \Delta_{Z_\bullet}(L|_Z).$$

Thus Proposition 2.1 is the explicit product form of Proposition 5.1. In the general nef conormal case no such product decomposition should be expected; the intersection polynomial D_t^n retains the Segre classes of $N_{Z/X}$.

References

[1] F. Ballaÿ, *Volume functions on blow-ups and Seshadri constants*, Proc. Amer. Math. Soc. **150** (2022), no. 5, 1925–1935.

[2] T. de Fernex, A. Küronya, and R. Lazarsfeld, *Higher cohomology of divisors on a projective variety*, Math. Ann. **337** (2007), 443–455.

[3] A. Küronya and V. Lozovanu, *Positivity of line bundles and Newton–Okounkov bodies*, Doc. Math. **22** (2017), 1285–1302.

[4] R. Lazarsfeld and M. Mustaţă, *Convex bodies associated to linear series*, Ann. Sci. Éc. Norm. Supér. (4) **42** (2009), 783–835.

[5] C. Li, M. Miao and K. Zhang, *The sharp volume gap for Kähler manifolds with positive Ricci curvature*, arXiv:2608.08193, 2026, preprint.

[6] K. Zhang, *On the optimal volume upper bound for Kähler manifolds with positive Ricci curvature (with an appendix by Y. Liu)*, Int. Math. Res. Not. IMRN 2022, no. 8, 6135–6156.