The Second-Volume Problem for Positive Einstein Four-Manifolds

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Abstract

Let (M^4, g) be a closed connected Einstein four-manifold normalized by $\mathrm{Ric}_g = 3g$. Bishop–Gromov comparison gives $\mathrm{Vol}(M, g) \leq \mathrm{Vol}(S^4) = 8\pi^2/3$, with equality only for the unit round sphere. The natural "second-volume" conjecture asserts that every non-round example satisfies $\mathrm{Vol}(M, g) \leq 2\pi^2$, with equality only for $(\mathbb{CP}^2, 2g_{\mathrm{FS}})$. The strict part of this statement is exactly the assertion $C_{A,1}(4) = 3/4$ for the positive-Einstein Anderson constant introduced by Honda and Mondino. The equality classification is logically stronger and is not encoded by the value of the constant. This note explains that distinction, records the strongest elementary reduction obtained from the four-dimensional curvature identities and Gursky's sharp half-Weyl estimate, and gives a detailed list of properties forced on any hypothetical example with volume at least that of $\mathbb{CP}^2$. In particular, for oriented manifolds the conjecture is proved whenever $\min\{b_2^+, b_2^-\} \leq 1$ (and non-orientable manifolds are already well below the threshold); any remaining counterexample must be simply connected, geometrically irreducible, non-complex and non-symplectic in either orientation, have $b_2^+, b_2^- \geq 2$, have isometry group of dimension at most two, and violate explicit pointwise chiral-curvature cones in both orientations. The first unresolved intersection-form block is $(b_2^+, b_2^-) = (2, 2)$, where the known sharp one-sided inequalities collapse back to the Bishop bound.

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1 The problem and its normalization

Throughout, manifolds are smooth, closed, and connected. We normalize a positive Einstein metric by

$$\mathrm{Ric}_g = 3g, \tag{1}$$

so the scalar curvature is $s_g = 12$. The unit round metric g_{S^4} has this normalization and

$$\mathrm{Vol}(S^4, g_{S^4}) = \frac{8\pi^2}{3}. \tag{2}$$

Bishop–Gromov comparison, together with the equality case (or Cheng's diameter rigidity), gives

$$\mathrm{Vol}(M, g) \leq \frac{8\pi^2}{3}, \tag{3}$$

and equality holds exactly for the unit round sphere [Bes87, Che75, HM26].

Let g_{FS} denote the Fubini–Study metric obtained from the Hopf fibration $S^1 \to S^5 \to \mathbb{CP}^2$, with each fibre a great circle. Then

$$\mathrm{Ric}(g_{\mathrm{FS}}) = 6g_{\mathrm{FS}}, \qquad \mathrm{Vol}(\mathbb{CP}^2, g_{\mathrm{FS}}) = \frac{\pi^2}{2}.$$

Consequently $2g_{\mathrm{FS}}$ satisfies (1) and

$$\mathrm{Vol}(\mathbb{CP}^2, 2g_{\mathrm{FS}}) = 2\pi^2 = \frac{3}{4}\mathrm{Vol}(S^4, g_{S^4}). \tag{4}$$

Kewei Zhang came to me and proposed the following problems.

Conjecture 1.1 (Strict second-volume gap)**.** If (M^4, g) satisfies (1) and $\mathrm{Vol}(M, g) > 2\pi^2$, then (M, g) is the unit round S^4.

Conjecture 1.2 (Rigidity at the second volume)**.** If (M^4, g) satisfies (1), is not round, and $\mathrm{Vol}(M, g) = 2\pi^2$, then

$$(M, g) \cong (\mathbb{CP}^2, 2g_{\mathrm{FS}}).$$

I spent one hour on Kewei's problems. Here's my response to him.

The combined conjecture says that $\mathbb{CP}^2$ is the unique second-largest positive Einstein four-manifold. It agrees with every currently known smooth example, but neither Theorem 1.1 nor Theorem 1.2 is known in full generality. Honda–Mondino explicitly ask for the relevant sharp constant and, more broadly, for a classification above half the sphere volume [HM26, Questions 3.14 and 5.12]. Recent work on mass and volume also does not close the global problem [GM25]. In what follows I will elaborate on these points.

Acknowledgments. This note grew out of my conversation with Kewei Zhang. Honestly speaking, I have no idea why he pushed me so hard on math problems. I do not have the intention to be a fields medalist. The author is partially supported by Kewei Zhang, because for unknown reasons he has generously paid 200$ just to chat with me.

2 Anderson's constants and the exact logical relationship

2.1 The positive-Einstein constant

Honda and Mondino use the following positive-Einstein analogue of Anderson's Ricci-flat volume-gap constant [And89, HM26].

Definition 2.1 (Positive-Einstein Anderson constant)**.** For $n \geq 2$, $C_{A,1}(n)$ is the infimum of all $\varepsilon > 0$ with the property that every closed Einstein n-manifold satisfying

$$\mathrm{Ric} = (n-1)g, \qquad \mathrm{Vol}(M, g) > \varepsilon \, \mathrm{Vol}(S^n, g_{S^n})$$

is isometric to the unit round sphere.

Equivalently,

$$C_{A,1}(n) = \sup \left\{ \frac{\mathrm{Vol}(M, g)}{\mathrm{Vol}(S^n)} : \mathrm{Ric} = (n-1)g, \ (M, g) \not\cong S^n \right\}. \tag{5}$$

The strict inequality in the definition is important: the numerical value of $C_{A,1}(n)$ records a supremum but does not record which metrics attain it.

The Fubini–Study computation (4) gives

$$C_{A,1}(4) \geq \frac{3}{4}. \tag{6}$$

On the other hand, the positive-Einstein gap theorem gives a non-explicit $\varepsilon_4 > 0$ such that volume at least $(1 - \varepsilon_4) \, \mathrm{Vol}(S^4)$ forces roundness [HM21, HM26]. Thus the present rigorous range is

$$\frac{3}{4} \leq C_{A,1}(4) < 1. \tag{7}$$

By (5), Theorem 1.1 is precisely the assertion

$$\boxed{C_{A,1}(4) = \frac{3}{4}.} \tag{8}$$

However, (8) alone does not imply Theorem 1.2: another disconnected Einstein component could in principle have exactly the same normalized volume $3/4$. Conversely, an equality classification at $2\pi^2$ would not by itself rule out a metric of larger volume. The two conjectures must therefore be audited separately.

2.2 The Ricci-flat constant and links of cones

The notation comes from the Ricci-flat gap problem. If (X^n, h) is complete, Ricci-flat, and has Euclidean volume growth, write

$$\mathrm{AVR}(X,h) = \lim_{r\to\infty} \frac{\mathrm{Vol}(B_r(x))}{\omega_n r^n}.$$

The constant $C_{A,0}(n)$ is the infimum of $\varepsilon > 0$ such that $\mathrm{AVR} > \varepsilon$ forces (X,h) to be Euclidean. Anderson's gap theorem shows $C_{A,0}(n) < 1$ [And89]. In dimension four,

$$C_{A,0}(4) = \frac{1}{2}, \tag{9}$$

with equality realized by the Eguchi–Hanson space [HM26].

The formal bridge between the two constants is the metric cone. If $C(Z^n)$ is the Ricci-flat cone over a smooth Einstein link satisfying $\mathrm{Ric}_Z = (n-1)g_Z$, then

$$\mathrm{AVR}(C(Z)) = \frac{\mathrm{Vol}(Z)}{\mathrm{Vol}(S^n)}. \tag{10}$$

This motivates Honda–Mondino's question whether $C_{A,0}(n+1) = C_{A,1}(n)$. They prove in the dimension relevant here that

$$C_{A,0}(5) \le C_{A,1}(4), \tag{11}$$

while the product of Eguchi–Hanson with $\mathbb{R}$ gives $C_{A,0}(5) \ge 1/2$ [HM26, Proposition 3.18]. Hence the second-volume conjecture would imply

$$\frac{1}{2} \le C_{A,0}(5) \le \frac{3}{4},$$

but would not determine $C_{A,0}(5)$ without an additional cone-link rigidity theorem.

2.3 Yamabe reformulation

Obata's theorem implies that an Einstein metric is a Yamabe metric in its conformal class [Bes87, Oba62]. For (1),

$$Y(M,[g]) = 12\sqrt{\mathrm{Vol}(M,g)}. \tag{12}$$

Consequently

$$\frac{\mathrm{Vol}(M,g)}{\mathrm{Vol}(S^4)} = \left(\frac{Y(M,[g])}{Y(S^4,[g_{S^4}])}\right)^2. \tag{13}$$

Thus $C_{A,1}(4)$ may also be viewed as the square of the largest normalized Yamabe constant among non-round positive Einstein conformal classes. The $\mathbb{CP}^2$ threshold corresponds to

$$Y(\mathbb{CP}^2, [g_{\mathrm{FS}}]) = 12\sqrt{2}\,\pi, \qquad Y(S^4, [g_{S^4}]) = 8\sqrt{6}\,\pi.$$

This is the natural language for the gap theorems of Gursky and of Akutagawa–Endo–Seshadri on S^4 [Gur00, AES19].

3 Known examples and their volumes

The following table uses the normalization $\mathrm{Ric} = 3g$. It lists the standard simply connected examples and two basic free quotients rather than claiming a classification. The Page and Chen–LeBrun–Weber (CLW) entries are numerical values; the remaining entries are exact [Bes87, Pag78, CLW08, HM14, ABBD26].

Manifold	metric	volume	ratio to S^4
S^4	round	$8\pi^2/3$	1
$\mathbb{CP}^2$	Fubini–Study	$2\pi^2$	$3/4$
$S^2 \times S^2$	equal product	$16\pi^2/9$	$2/3$
$\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$	Page	$\approx 1.6984\pi^2$	≈ 0.6369
$\mathbb{CP}^2 \# 2\overline{\mathbb{CP}}^2$	CLW	$\approx 1.4949\pi^2$	≈ 0.5606
$\mathbb{RP}^4$	round quotient	$4\pi^2/3$	$1/2$
$S^2 \times S^2 / \langle(-1,-1)\rangle$	product quotient	$8\pi^2/9$	$1/3$
$\mathbb{CP}^2 \# k\overline{\mathbb{CP}}^2$, $3 \leq k \leq 8$	Kähler–Einstein del Pezzo	$2\pi^2(9-k)/9$	$(9-k)/12$

The exact Kähler formula is worth recording. If a complex surface carries a Kähler–Einstein metric with $\mathrm{Ric} = 3g$, then its Ricci form satisfies $\rho = 3\omega$ and $[\rho] = 2\pi c_1(M)$. Hence

$$[\omega] = \frac{2\pi}{3} c_1(M), \qquad \mathrm{Vol}(M,g) = \frac{1}{2}[\omega]^2 = \frac{2\pi^2}{9} c_1^2(M). \tag{14}$$

For a smooth del Pezzo surface, $c_1^2 \leq 9$, with degree nine only for $\mathbb{CP}^2$; if the surface is not $\mathbb{CP}^2$, then $c_1^2 \leq 8$, giving

$$\mathrm{Vol}(M,g) \leq \frac{16\pi^2}{9} < 2\pi^2. \tag{15}$$

Existence, uniqueness, and degeneration of the relevant Kähler–Einstein metrics are treated in [BM87, Tia90, OSS16]. The two exceptional del Pezzo diffeotypes with $k = 1, 2$ carry the Page and CLW Einstein metrics, which are conformally Kähler but not Kähler [Der83, CLW08, LeB12].

Thus $\mathbb{CP}^2$ is indeed second largest among the standard presently known examples, and the basic smooth finite quotients lie well below it. Recent physics literature carefully phrases this as a statement about the known positive Einstein metrics [ABBD26]; it is not a global classification theorem.

4 Four-dimensional identities and a sharp topological reduction

4.1 Orientability and the fundamental group at the threshold

Lemma 4.1. *If* (M^4, g) *satisfies* $\mathrm{Ric} = 3g$ *and* $\mathrm{Vol}(M, g) \geq 2\pi^2$*, then* M *is orientable and simply connected.*

Proof. If M were non-orientable, its oriented double cover would have volume at least $4\pi^2$, contradicting (3). Myers' theorem makes $\pi_1(M)$ finite. If $d = |\pi_1(M)|$, the universal cover has volume dV, so Bishop comparison gives

$$d(2\pi^2) \leq dV \leq \frac{8\pi^2}{3}.$$

Thus $d \leq 4/3$ and hence $d = 1$. □

This elementary observation is already strong: finite quotients, including $\mathbb{RP}^4$, are irrelevant at and above the $\mathbb{CP}^2$ threshold.

4.2 Gauss–Bonnet, signature, and the two chiral energies

Fix an orientation and write

$$p = b_2^+(M), \qquad q = b_2^-(M).$$

By Theorem 4.1, at the threshold one has

$$\chi(M) = 2 + p + q, \qquad \tau(M) = p - q. \tag{16}$$

Let

$$A = \int_M |W^+|^2 \, d\mu, \qquad B = \int_M |W^-|^2 \, d\mu.$$

For an Einstein four-manifold with scalar curvature 12, the Chern–Gauss–Bonnet and Hirzebruch signature formulae are

$$8\pi^2\chi(M) = A + B + 6V, \tag{17}$$
$$12\pi^2\tau(M) = A - B. \tag{18}$$

Equivalently,

$$A + 3V = 2\pi^2(2\chi + 3\tau) = 2\pi^2(4 + 5p - q), \tag{19}$$
$$B + 3V = 2\pi^2(2\chi - 3\tau) = 2\pi^2(4 - p + 5q). \tag{20}$$

Here $W^\pm : \Lambda^\pm \to \Lambda^\pm$ acts by $(W^\pm\eta)_{ij} = \frac{1}{2}W_{ij}{}^{kl}\eta_{kl}$, and our norm convention is

$$|W^\pm|^2 = \mathrm{tr}\big((W^\pm)^2\big). \tag{21}$$

This is one quarter of the raw $(0, 4)$-tensor contraction used in some sources. See [AHS78, Bes87, Hit74].

4.3 Gursky's half-Weyl inequality

Einstein metrics have $\delta W^{\pm} = 0$. Gursky's sharp theorem, applied in either orientation, gives the following alternative [Gur98, Gur00, LeB22]:

$$W^{\pm} \equiv 0 \quad \text{or} \quad \int_M |W^{\pm}|^2 \, d\mu \geq 6V. \tag{22}$$

In the nonzero case this is equivalent, using (19)–(20), to

$$V \leq \frac{2\pi^2}{9}(2\chi \pm 3\tau). \tag{23}$$

The equality case has a minor but important covering subtlety. If the relevant $b_2^{\pm}$ is positive, equality is equivalent to the conformal class containing a positive Kähler–Einstein metric; without this cohomological hypothesis the Kähler metric may exist only on a double cover. The hypothesis is satisfied in every equality application below. Since two conformally related Einstein metrics on a closed manifold are homothetic unless the metric is round, this becomes Kähler–Einstein rigidity in the present non-round applications [Bes87, Gur98].

If $W^{\pm} \equiv 0$, Hitchin's theorem classifies the positive half-conformally-flat Einstein universal cover as S^4 or $\mathbb{CP}^2$ with its standard metric [Hit74].

4.4 A proved second-volume theorem

Theorem 4.2. *Let (M^4, g) be an oriented, non-round Einstein manifold with $\mathrm{Ric} = 3g$. If*

$$\min\{b_2^+(M), b_2^-(M)\} \leq 1,$$

then

$$\mathrm{Vol}(M, g) \leq 2\pi^2.$$

If equality holds, then

$$(M, g) \cong (\mathbb{CP}^2, 2g_{\mathrm{FS}}).$$

Proof. It is enough to consider $V \geq 2\pi^2$, since otherwise there is nothing to prove. By Theorem 4.1, M is oriented and simply connected. Reverse the orientation if necessary so that $p = b_2^+ \leq 1$.

If $W^+ \equiv 0$, Hitchin's theorem gives the round sphere or standard $\mathbb{CP}^2$; the round case is excluded. If $W^+ \not\equiv 0$, combine (19) with (22):

$$9V \leq 2\pi^2(4 + 5p - q) \leq 2\pi^2(4 + 5p) \leq 18\pi^2.$$

Thus $V \leq 2\pi^2$.

Equality in the nonzero branch forces $p = 1$, $q = 0$, and equality in Gursky's estimate. Hence g is Kähler–Einstein after a homothety, and

$$c_1^2 = 2\chi + 3\tau = 9.$$

The corresponding del Pezzo surface is $\mathbb{CP}^2$, and Kähler–Einstein uniqueness identifies the metric with the normalized Fubini–Study metric [BM87]. The half-conformally-flat branch gives the same conclusion. □

The theorem includes all definite intersection forms and many more cases. In particular, a non-round Einstein metric on a rational homology four-sphere would satisfy the much stronger bound $V \leq 8\pi^2/9$. If M is non-orientable, its oriented double cover and Bishop comparison give $V \leq 4\pi^2/3$, so non-orientable manifolds are also far below the threshold.

5 Why equality at $2\pi^2$ is a separate problem

The proof of Theorem 4.2 characterizes equality only because the topological term $2\chi + 3\tau$ is at most nine. It gives no global equality mechanism when both b_2^+ and b_2^- are at least two.

The first unresolved block is

$$(p, q) = (2, 2), \qquad \chi = 6, \qquad \tau = 0. \tag{24}$$

Here (19)–(20) become

$$A = B = 24\pi^2 - 3V. \tag{25}$$

At the conjectural equality volume $V = 2\pi^2$,

$$A = B = 18\pi^2, \tag{26}$$

whereas Gursky's lower bound is only $A, B \geq 12\pi^2$. Thus no equality case of the known sharp estimate is activated. If $V > 2\pi^2$, applying the same estimate to (25) yields only

$$24\pi^2 - 3V \geq 6V \quad \Longrightarrow \quad V \leq \frac{8\pi^2}{3},$$

which is exactly the Bishop bound. This is the precise analytic bottleneck.

There are at least three conceptually different ways a complete equality theorem could be obtained:

1. derive topology from volume, for example prove that $V \geq 2\pi^2$ forces $\min\{b_2^+, b_2^-\} \leq 1$;
2. prove a genuinely joint inequality coupling W^+ and W^-, rather than applying the one-sided estimate independently;
3. find a new equality-sensitive invariant (mass, a gauge-theoretic invariant, or a canonical harmonic form) that excludes the high-topology block even though both Gursky inequalities are strict.

None of these mechanisms is currently available in the required generality.

There is also no elementary variational shortcut. If g_t is a differentiable family satisfying $\mathrm{Ric}(g_t) = 3g_t$ and $h = \dot{g}$, then differentiating $s_{g_t} = 12$ and integrating the scalar-curvature variation gives

$$0 = \int_M \dot{s}\, d\mu = -3 \int_M \mathrm{tr}_g\, h\, d\mu.$$

Therefore

$$\frac{d}{dt} \mathrm{Vol}(M, g_t) = \frac{1}{2} \int_M \mathrm{tr}_g\, h\, d\mu = 0. \tag{27}$$

Normalized volume is locally constant along Einstein paths. Hence a second-largest volume is an ordering of disconnected Einstein components, not a usual local maximum problem for the Einstein–Hilbert functional. Local rigidity or stability of $\mathbb{CP}^2$ cannot by itself exclude a separate component at the same or larger volume [Koi82, SS25].

6 The forced profile of a counterexample

Assume from now on that (M^4, g) is non-round, $\mathrm{Ric} = 3g$, and

$$V \geq 2\pi^2, \tag{28}$$

but (M, g) is not the normalized Fubini–Study $\mathbb{CP}^2$. If the inequality is strict, this is a counterexample to Theorem 1.1; at equality it is a counterexample to Theorem 1.2.

6.1 Topology

The following constraints are rigorous consequences of the preceding identities and standard four-manifold theory.

T1. Simply connected and orientable. This is Theorem 4.1. In particular $b_1 = 0$.

T2. Both halves of the intersection form are large. By Theorem 4.2,

$$p = b_2^+ \geq 2, \qquad q = b_2^- \geq 2. \tag{29}$$

Thus $\chi = 2 + p + q \geq 6$.

T3. Sharper integer geography. Both W^+ and W^- are nonzero, and equality in Gursky's estimate would force a del Pezzo orientation, contrary to (29). Hence the two Gursky inequalities are strict. From (19)–(20),

$$4 + 5p - q \geq 10, \qquad 4 - p + 5q \geq 10, \tag{30}$$

or equivalently

$$q \leq 5p - 6, \qquad p \leq 5q - 6.$$

T4. No symplectic structure in either orientation. If one orientation were symplectic, its b_2^+ would exceed one. Taubes' nonvanishing theorem would then give a nonzero Seiberg–Witten invariant, whereas the positive scalar-curvature metric g forces all such invariants to vanish. This contradiction applies in both orientations [Wit94, Tau94, LeB09].

T5. No complex structure. LeBrun's topology theorem says that a smooth four-manifold admitting both a complex structure and a nonnegative-Einstein metric also admits a symplectic structure (the structures and the metric need not be compatible) [LeB09]. This directly contradicts the preceding item. Thus the underlying smooth manifold cannot be a compact complex surface in either orientation.

T6. If spin, then the signature vanishes. For a spin manifold, the Lichnerowicz formula and positive scalar curvature give $\widehat{A}(M) = 0$. In dimension four, $\widehat{A}(M) = -\tau(M)/8$, so

$$\tau = 0, \qquad p = q \geq 2. \tag{31}$$

The smallest even intersection-form possibility is $2H$. In the non-spin case the first odd possibility is $I_{2,2}$ [LM89, Fre82, Don83].

The block $(p, q) = (2, 2)$ is therefore not artificial: both the even form $2H$ and the odd form $I_{2,2}$ survive all elementary index and Hitchin–Thorpe tests.

6.2 Global geometry and symmetry

G1. Geometrically irreducible. Because M is simply connected, a reducible metric would split globally by de Rham. Positive Einstein factors in dimension four can only give the equal product $S^2 \times S^2$, whose volume is $16\pi^2/9 < 2\pi^2$. Hence the hypothetical metric is irreducible.

G2. Low symmetry. Broder's 2026 classification shows that every closed Einstein four-manifold with isometry group of dimension at least three is, up to cover and scale, flat or one of S^4, $\mathbb{CP}^2$, $S^2 \times S^2$, and the Page metric [Bro26]. Therefore

$$\dim \mathrm{Isom}(M, g) \leq 2. \tag{32}$$

G3. No singular-orbifold degeneration at high volume. Let $\mathcal{O}$ be a connected closed effective Einstein four-orbifold with $\mathrm{Ric} = 3g$, and suppose that p is a genuine singular point with effective local group $\Gamma \neq 1$. Orbifold Bonnet–Myers gives $\mathrm{diam}(\mathcal{O}) \leq \pi$. Applying orbifold Bishop–Gromov at p, and taking first $r = \pi$ and then the small-radius density, gives

$$\frac{\mathrm{Vol}(\mathcal{O})}{\mathrm{Vol}(S^4)} \leq \lim_{r \downarrow 0} \frac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}(B_r^{S^4})} = \frac{1}{|\Gamma|} \leq \frac{1}{2}. \tag{33}$$

By volume convergence, no noncollapsed Gromov–Hausdorff convergent sequence of smooth metrics with $\mathrm{Ric} = 3g$ and volume at least $3/4$ of the sphere can have such a singular orbifold limit. Honda–Mondino make this quantitative: for each $\delta > 0$, the class with $\mathrm{Vol} \geq (1/2 + \delta)\,\mathrm{Vol}(S^4)$ is compact in the smooth topology [HM26, Proposition 3.19]. Ozuch excludes spherical and hyperbolic orbifolds with the simplest singularities as smooth Einstein limits; Huang–Ozuch prove isolation for all spherical and hyperbolic Einstein four-orbifolds; and LeBrun–Ozuch prove conditional rigidity for Hermitian orbifold limits when every bubble is anti-self-dual [Ozu24, HO25, LO26].

G4. Uniform high-volume regularity. The preceding compactness supplies uniform curvature, injectivity-radius, and derivative bounds (after diffeomorphism) for the entire class $V \geq 2\pi^2$. A counterexample is therefore not expected to arise through a small-scale orbifold gluing limit; it must be a smooth component of the positive Einstein moduli problem.

6.3 Chiral Weyl curvature

W1. Both Weyl halves are nonzero and have large energy. One has

$$A = 2\pi^2(4 + 5p - q) - 3V > 6V, \tag{34}$$
$$B = 2\pi^2(4 - p + 5q) - 3V > 6V. \tag{35}$$

In the first block $p = q = 2$, and for $2\pi^2 \leq V \leq 8\pi^2/3$, both energies lie between $16\pi^2$ and $18\pi^2$. Thus no small-L^2 Weyl pinching argument begins in this block.

W2. Both canonical curvature blocks fail positivity somewhere. For $s = 12$, the curvature operators of the Levi-Civita connections on $\Lambda^\pm$ are

$$\mathcal{R}_\pm = \mathrm{Id} + W^\pm. \tag{36}$$

Suppose $\mathcal{R}_+$ were positive definite everywhere. If $w_1 \leq w_2 \leq w_3$ are the trace-free eigenvalues of W^+, then $w_1 > -1$ and elementary algebra gives $|W^+|^2 < 6$ pointwise unless $W^+ = 0$. This contradicts $A \geq 6V$. Hence a counterexample must satisfy

$$\lambda_{\min}(\mathrm{Id} + W^+) \leq 0 \quad \text{somewhere}, \tag{37}$$

and the same statement holds for W^-. The threshold is naturally degenerate: for normalized $\mathbb{CP}^2$ in its complex orientation,

$$W^+ = \operatorname{diag}(2,-1,-1), \qquad \mathrm{Id} + W^+ = \operatorname{diag}(3,0,0).$$

Thus positivity of the canonical connection is too strong to detect the conjectured equality metric. See [FKP14] for the definite connection framework and its limitations.

W3. Both orientations enter an explicit bad determinant cone. Set

$$c_0 = \frac{5\sqrt{2}}{21\sqrt{21}}.$$

LeBrun proves that a compact metric with $\delta W^+ = 0$, $W^+ \not\equiv 0$, and

$$\det(W^+) \geq -c_0|W^+|^3 \quad \text{everywhere}$$

actually has $\det(W^+) > 0$ and, after at most a double cover, is conformally Kähler [LeB21, Theorem C]. Since the present M is simply connected and cannot be del Pezzo, there must be a point with

$$\det(W^+) < -c_0|W^+|^3. \tag{38}$$

After reversing orientation, there must also be a (possibly different) point with

$$\det(W^-) < -c_0|W^-|^3. \tag{39}$$

W4. No half-conformal, Hermitian, or conformally Kähler description. Half-conformal flatness is excluded by Hitchin; a positive Einstein Hermitian metric is one of the known conformally Kähler del Pezzo metrics [Hit74, Der83, LeB12]. Hence any counterexample must be genuinely non-Hermitian and non-chiral.

7 What the main existing methods do and do not prove

7.1 Kähler, Hermitian, symplectic, and half-conformal cases

The conjecture is completely settled in each of the following regimes:

- positive Kähler–Einstein surfaces, by the Chern-number volume formula (14);
- positive Einstein Hermitian surfaces, by the Derdziński–LeBrun–Chen–Weber theory [Der83, CLW08, LeB12];
- manifolds carrying an unrelated symplectic or complex structure, by the two-orientation Seiberg–Witten argument and LeBrun's complex–symplectic topology theorem [LeB09];
- half-conformally-flat positive Einstein metrics, by Hitchin [Hit74];
- $\min\{b_2^+, b_2^-\} \leq 1$, by Theorem 4.2;
- $\dim \operatorname{Isom} \geq 3$, by Broder's classification [Bro26].

None of these hypotheses is presently known to follow from the volume lower bound alone.

7.2 Seiberg–Witten and Spin^c methods

Gauge theory is extremely effective when a monopole class is available. For example, Gursky–LeBrun type estimates bound Yamabe invariants and Einstein volumes using the self-dual projection of a characteristic class [GL99, LeB96, LeB09]. The difficulty is structural: the forced counterexample has $b_2^+ > 1$ in both orientations and admits a positive scalar-curvature metric, so its Seiberg–Witten invariants vanish. It also cannot be symplectic, so Taubes' theorem supplies no canonical basic class. A purely lattice-theoretic choice of a characteristic element cannot replace the missing monopole information because its self-dual projection depends on the unknown period point of g.

The recent Vafa–Witten approach yields the same sharp topological half-Weyl lower bound under the additional existence of an irreducible Vafa–Witten solution, but such a solution is not known to exist for every positive Einstein four-manifold [HZ26]. It therefore does not close the unconditional gap.

7.3 Definite connections and twistor geometry

The decomposition (36) turns four-dimensional Einstein geometry into an $\mathrm{SO}(3)$ gauge problem. When a chiral curvature block is definite, the associated twistor sphere bundle carries a natural symplectic structure and powerful classification questions become available [FKP14]. But this symplectic form lives on the six-dimensional twistor space, not on M, and the general classification of positive definite connections remains conjectural. More decisively, the $\mathbb{CP}^2$ equality metric already makes $\mathrm{Id} + W^+$ degenerate. Volume information does not prevent sign changes of the curvature block; the Page metric provides an explicit known example of such behaviour [FKP14].

7.4 Compactness, gluing, and mass

Anderson–Bando–Kasue–Nakajima compactness describes degeneration of Einstein four-manifolds by orbifold limits and Ricci-flat ALE bubbles [And89, And90, Ban90, BKN89, CN15]. At the present volume threshold, however, (33) prevents a singular orbifold limit. Recent work gives isolation or desingularization results in specific spherical, hyperbolic, and Hermitian settings; in the Hermitian result the anti-self-dual-bubble hypothesis is essential [Ozu24, HO25, LO26]. Consequently the most obvious high-volume gluing mechanism cannot produce a counterexample.

Gursky–Malchiodi associate an asymptotically flat scalar-flat metric to the Green's function of the conformal Laplacian and prove mass–volume inequalities and mass gap theorems characterizing S^4 and $\mathbb{CP}^2$ under additional mass hypotheses [GM25]. These results are directly relevant to equality rigidity, but no known universal control of the mass converts them into $C_{A,1}(4) = 3/4$.

7.5 Einstein moduli and stability

The local premoduli space of Einstein metrics is a finite-dimensional real analytic set [Koi82, Bes87, SS25]. Together with Honda–Mondino compactness, this makes the high-volume class analytically well-behaved. Nevertheless, (27) shows why local deformation theory cannot order different components by volume. Moreover, notions of Einstein–Hilbert, Ricci-flow, or Perelman-entropy stability do not follow from being a globally large critical value. In particular, local rigidity of the round sphere or of Fubini–Study does not rule out a distant high-topology component.

8 Concrete targets for further work

The reductions above suggest several precise statements, any one of which would materially advance the problem.

Question 8.1 (Topology from volume)**.** Does $\mathrm{Ric} = 3g$ and $V \geq 2\pi^2$ force $\min\{b_2^+, b_2^-\} \leq 1$?

An affirmative answer, combined with Theorem 4.2, proves both the strict gap and equality rigidity.

Question 8.2 (Coupled chiral inequality)**.** Is there a sharp inequality involving both W^+ and W^- which improves the two independent Gursky bounds when $b_2^+, b_2^- \geq 2$?

Any useful version must exclude (26); one-sided L^2 pinching cannot do so.

Question 8.3 (High-volume harmonic geometry)**.** Can the volume threshold force a nowhere-zero harmonic two-form, positivity of a chiral curvature block, or a weaker condition that still produces a symplectic structure on M?

Such a result would contradict the non-symplectic requirement in both orientations. It must allow the semidefinite block of $\mathbb{CP}^2$, so ordinary strict definiteness is not the right endpoint condition.

Question 8.4 (Equality-sensitive mass)**.** Can the Green's-function mass in [GM25] be controlled from above or below using only $V \geq 2\pi^2$ and the Einstein equation, with a sharp equality case at $\mathbb{CP}^2$?

This would directly address the logical gap between the numerical Anderson constant and classification of its maximizers.

9 Conclusion

The second-volume conjecture has two layers:

$$\begin{aligned} \text{strict gap:} \quad & C_{A,1}(4) = \frac{3}{4}, \\ \text{endpoint rigidity:} \quad & V = 2\pi^2,\ M \ncong S^4 \Longrightarrow (M, g) \cong (\mathbb{CP}^2, 2g_{\mathrm{FS}}). \end{aligned}$$

The first is exactly Honda–Mondino's Anderson-constant problem in dimension four; the second is additional. The combined statement is true for all known examples and for several broad, rigorously classified subclasses. The sharp four-dimensional identities prove it whenever one chiral Betti number is at most one.

What remains is not an unexamined equality case. It is a specific high-topology regime beginning at $(b_2^+, b_2^-) = (2, 2)$, in which both Weyl halves carry substantial energy and all standard one-sided sharp inequalities have room to spare. Any counterexample must be smooth, simply connected, irreducible, low-symmetry, non-complex, non-symplectic in either orientation, and pointwise far from the conformally Kähler Weyl cone in both chiralities. No such positive Einstein four-manifold is currently known.

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