BOUNDING THE VOLUME OF KÄHLER MANIFOLDS WITH POSITIVE RICCI CURVATURE

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Abstract. This is a continuation of [57]. Using Calabi ansatz and Fano index, we give several optimal volume upper bounds for polarized Kähler manifolds with positive Ricci curvature. Along the way we obtain some new characterizations of the complex projective space.

1. Introduction

Let (X, g) be an m-dimensional Riemannian manifold such that

$$\mathrm{Ric}\,(g) \geq (m-1)g.$$

Then the famous Bishop–Gromov volume comparison says that

$$\mathrm{Vol}(X, g) \leq \mathrm{Vol}(S^m, g_{S^m}),$$

and the equality holds if and only if (X, g) is isometric to the standard m-sphere S^m. However, suppose in addition that X has a complex structure J such that (X, g, J) is Kähler, then Liu [40] shows that this volume upper is *never sharp* (unless $X = \mathbb{P}^1$), in the sense that there exists a dimensional gap $\epsilon(n) > 0$ such that

$$\mathrm{Vol}(X, g) \leq \mathrm{Vol}(S^m, g_{S^m}) - \epsilon(n).$$

This distinguishes the Kählerian geometry from the Riemannian case (see also Li–Wang [38] for related discussions using holomorphic bisectional curvatures). So it is natural to ask what the optimal volume upper bound should be. In [57] the author obtained the optimal upper bound for compact Kähler manifolds with positive Ricci curvature, which is stated as follows:

Theorem 1.1 ([57]). *Let (X, ω) be a Kähler manifold of dimension n that satisfies*

$$\mathrm{Ric}\,(\omega) \geq (n+1)\omega,$$

then

$$\mathrm{Vol}(X, \omega) \leq \mathrm{Vol}(\mathbb{P}^n, \omega_{FS}),$$

and the equality holds if and only if (X, ω) is biholomorphically isometric to $(\mathbb{P}^n, \omega_{FS})$. Here $\mathbb{P}^n$ denotes the complex projective space and ω_{FS} is the Fubini–Study metric on $\mathbb{P}^n$ such that $\int_{\mathbb{P}^n} \omega_{FS}^n = (2\pi)^n$.

The proof of the above result mainly uses the greatest Ricci lower bound and its relation to the δ-invariant. The essential point is to reformulate the Ricci curvature condition using twisted Kähler–Einstein equations, which gives a lower bound of the analytic δ-invaraint introduced in [58]. This further implies a lower bound of the algebraic δ-invariant by [58, Proposition 4.5] (see also [59]). And then Fujita's estimate in [16] gives the optimal volume upper bound of the cohomology class.

In this article we will mainly focus on the case when ω lies in the first Chern class of some ample line bundle. As we shall see, one can use Calabi ansatz to derive an optimal volume upper bound for the line bundle. Our methods is related to Sasakian geometry and also gives some partial answer for the ordinary double point (ODP) conjecture [48, Conjecture 1.2]. Note that Calabi ansatz has also been utilized in the recent joint work [60] of the author to explicitly compute δ-invaraints of projective bundles.

To state our results more precisely, let us fix some notions. Throughout this article X denotes an n-dimensional Fano manifold, meaning that its anti-canonical line bundle $-K_X$ is ample. For any line bundle L on X one can define its index $I(L)$ to be

$$I(L) := \max\{k \in \mathbb{N} \mid \exists \text{ line bundle } H \text{ on } X \text{ s.t. } L = kH\} \tag{1.1}$$

When $L = -K_X$, we put

$$I(X) := I(-K_X), \tag{1.2}$$

which is called the *Fano index* of X.

In what follows, L is always assumed to be *ample*. In this case, one can naturally define the greatest Ricci lower bound $\beta(X, L)$ by

$$\beta(X, L) := \sup\{\mu > 0 \mid \exists \text{ Kähler form } \omega \in 2\pi c_1(L) \text{ s.t. } \mathrm{Ric}\,(\omega) \geq \mu\omega\}. \tag{1.3}$$

Note that, by the Calabi–Yau theorem, given any Kähler form $\alpha \in 2\pi c_1(X)$, one can always find $\omega_0 \in 2\pi c_1(L)$ such that $\mathrm{Ric}\,(\omega_0) = \alpha > 0$. By compactness of X we see $\mathrm{Ric}\,(\omega_0) \geq \epsilon\omega_0$ for some $\epsilon > 0$. So $\beta(X, L)$ is always a positive number. On the other hand, $\beta(X, L)$ is naturally bounded from above by the Seshadri constant

$$\epsilon(X, L) := \sup\{\mu > 0 \mid -K_X - \mu L \text{ is nef}\}. \tag{1.4}$$

Thus we always have

$$0 < \beta(X, L) \leq \epsilon(X, L). \tag{1.5}$$

Remark 1.2. In the special case $L = -K_X$,

$$\beta(X) := \beta(X, -K_X)$$

is the usual greatest Ricci lower bound. This invariant was the topic of Tian's article [55] although it was not explicitly defined there, but was first explicitly defined by Rubinstein in [44, (32)], [45, Problem 3.1] and was later further studied by Székelyhidi [52], Li [33], Song–Wang [49], Cable [6], et al.

Our goal is to bound the volume $\mathrm{Vol}(L)$. Here $\mathrm{Vol}(L) := (c_1(L))^n$ denotes the volume of the Kähler class $c_1(L)$. Deriving volume upper bounds for ample line bundles is of interest by itself, as it is closely related to the boundedness problem in algebraic geometry. For instance, by bounding $(-K_X)^n$ from above, it was shown by Kollár–Miyaoka–Mori [29] that n-dimensional Fano manifolds form a bounded family. However to the author's knowledge, a *sharp* volume upper bound for $-K_X$ is still missing in the literature. It was once believed that one always has

$$(-K_X)^n \leq (n+1)^n \tag{1.6}$$

with equality if and only if $X = \mathbb{P}^n$. But it turns out that one can easily disprove this by looking at projective bundles over Fano manifolds (see [2, 13]). However when $\mathrm{rk}\,\mathrm{Pic}(X) = 1$, it is still an open problem whether (1.6) holds or not (cf.

[28, 21]). Note that when X is K-semistable, (1.6) has been recently estabilished by K. Fujita [16] (see also Liu [36] for volume upper bounds of $\mathbb{Q}$-Fano varieties).

For general ample line bundles, one can easily bound $\mathrm{Vol}(L)$ in terms of the greatest Ricci lower bound. Indeed, pick $\omega \in 2\pi c_1(L)$ such that

$$\mathrm{Ric}\,(\omega) \geq \mu\omega.$$

Applying Bishop–Gromov theorem to (X, ω), one can quickly derive (by letting $\mu \to \beta(X, L)$)

$$\beta(X, L)^n \mathrm{Vol}(L) \leq \frac{2^{n+1}(n!)^2(2n-1)^n}{(2n)!}. \tag{1.7}$$

However, as alluded to at the very beginning, this bound is not sharp in the Kählerian world. To get sharp bound, one needs to exploit the Kähler condition. Our first main result is the following

Theorem 1.3. *Let L be an ample line bundle on a Fano manifold X. Then one has*

$$I(L)\beta(X, L)^{n+1}\mathrm{Vol}(L) \leq (n+1)^{n+1}. \tag{1.8}$$

Remark 1.4. *Notice that $I(L)\beta(X, L)^{n+1}\mathrm{Vol}(L)$ is scaling invariant, so the conclusion of Theorem 1.3 also holds for any ample $\mathbb{Q}$-line bundle.*

The bound (1.8) is sharp, in the sense that $X = \mathbb{P}^n$ achieves the equality. An easy consequence of Theorem 1.3 is the following

Corollary 1.5. *Let L be an ample line bundle on a Fano manifold X. Then one has*

$$I(L)\beta(X, L) \leq n + 1.$$

Let us also say a few words about the lower bound of $I(X)\beta(X, L)$. When $L = -K_X$, by utilizing the boundedness of Fano manifolds [29] and the α-invariant of Tian [54], it follows that $I(X)\beta(X) \geq c(n) > 0$ for some dimensional constant $c(n)$. However for general ample line bundles, there is no universal positive lower bound for $I(L)\beta(X, L)$, as one can easily find a sequence L_i in the ample cone with $I(L_i) = 1$ and $\epsilon(X, L_i) \to 0$, so that $\beta(X, L_i) \to 0$.

When $L = -K_X$ and $\beta(X) = 1$ (i.e., X is K-semistable [34]), (1.8) reduces to the inequality:

$$I(X)(-K_X)^n \leq (n+1)^{n+1},$$

which was derived in [19, (2.22)] under the assumption that X is Kähler–Einstein using Sasakian geometry. In fact we will use a similar method to prove (1.8). More precisely, the strategy is to look at the affine cone

$$V := \mathrm{Spec} \bigoplus_{m \geq 0} H^0(X, mL).$$

Using Calabi ansatz, one can construct a cone metric on V with non-negative Ricci curvature. Then we apply the comparison theorem of Bishop–Gromov to the link of this metric cone to get (1.8).

Theorem 1.6. *Let L be an ample $\mathbb{Q}$-line bundle on a Fano manifold X. Then one has*

$$\begin{cases} I(L)\beta(X,L)^{n+1}\mathrm{Vol}(L)=(n+1)^{n+1}, & \text{when } X\cong\mathbb{P}^n;\\ I(L)\beta(X,L)^{n+1}\mathrm{Vol}(L)=2n^{n+1}, & \text{when } X\cong Q\subset\mathbb{P}^{n+1} \text{ is a smooth quadric;}\\ I(L)\beta(X,L)^{n+1}\mathrm{Vol}(L)<n(n+1)^{n}, & \text{otherwise.}\end{cases}$$

The proof of this result is in fact a combination of Theorem 3.1, (3.2) and some classical results in algebraic geometry (cf. [18, 23, 26]). As we shall see, Theorem 1.6 is also related to the ordinary double point (ODP) conjecture [48, Conjecture 1.2].

Organization. The rest of this paper is organized as follows. In Section 2 we review a classical setting where one can apply the Calabi ansatz and then construct a family of metric cones to prove Theorem 1.3 and Corollary 1.5. In Section 3, the equality case of (1.8) is investigated and Theorem 1.6 is proved. In Section 4 we connect our discussions to the normalized volume introduced by Li [36].

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2. Calabi ansatz and metric cones

In this section we use Calabi ansatz to prove Theorem 1.3.

2.1. Calabi ansatz on the total space of line bundles.

For readers' convenience, let us review a well-studied and powerful construction, pioneered by Calabi [7, 8], which can effectively produce various explicit examples of canonical metrics in Kähler geometry. The idea is to work on complex manifolds with certain symmetries so that one can reduce geometric PDEs to simple ODEs. This approach is often referred to as the Calabi ansatz in the literature, which has been studied and generalzed to different extent by many authors; see e.g., [20] for some general discussions and historical overviews.

For our purpose, we will work on the total space of line bundles over Kähler manifolds. The goal is to construct canonical metrics on this space. Our computation will follow the exposition of [53, Section 4.4]. See also [27, 47, 56] and the references therein for similar treatment.

Let (X,ω) be an n-dimensional compact Kähler manifold, where ω is a Kähler form on X. Let $L\to X$ be a holomorphic line bundle equipped with a smooth Hermitian metric h such that its curvature form R_h satisfies

$$R_h:=\sqrt{-1}\partial\bar\partial\log h^{-1}=\lambda\omega \tag{2.1}$$

for some constant $\lambda\neq 0$. Let

$$L^{-1}\xrightarrow{\pi}X$$

be the dual bundle of L. whose zero section will be denoted by E_0 (so E_0 is a copy of X sitting inside the total space L^{-1}). In the following we will construct a Kähler metric on $L^{-1}\backslash\{E_0\}$.

The idea is to make use of the fiberwise norm on L^{-1} induced by h^{-1}. We put

$$s(t):=\log||t||^2=\log h^{-1}(t,t),\ \text{for } t\in L^{-1}\backslash\{E_0\}.$$

So s is a globally defined function on $L^{-1}\backslash\{E_0\}$. The goal is to construct a Kähler metric η on $L^{-1}\backslash\{E_0\}$ of the form

$$\eta = \sqrt{-1}\partial\bar{\partial} f(s), \tag{2.2}$$

where f is a function to be determined.

We will carry out the computation locally. Choose $p \in X$ and let $\big(U, z = (z_1, ..., z_n)\big)$ be a local coordinate system around p such that ω can be expressed by a Kähler potential:

$$\omega = \sqrt{-1}\partial\bar{\partial}(P(z)), \tag{2.3}$$

where

$$P(z) = |z|^2 + O(|z|^4).$$

Moreover we may assume that L^{-1} is trivialized over U by a nowhere vanishing holomorphic section $\sigma \in \Gamma(U, L^{-1})$ such that

$$||\sigma||^2_{h^{-1}} = h^{-1}(\sigma, \sigma) = e^{\lambda P(z)}.$$

Under this trivialization, we have an identification:

$$\pi^{-1}(U) \cong U \times \mathbb{C}. \tag{2.4}$$

Let w be the holomorphic coordinate function in the fiber direction. So we have

$$s = \log(|w|^2 e^{\lambda P(z)}) \text{ on } U \times \mathbb{C}^*. \tag{2.5}$$

Such a choice of coordinates has the advantage that, on the fiber $\pi^{-1}(p)$ over p, one has

$$\partial P(z) = \overline{\partial} P(z) = 0. \tag{2.6}$$

So direct computation gives

$$\eta = \sqrt{-1}\partial\bar{\partial} f(s) = \lambda f' \pi^*\omega + f'' \frac{\sqrt{-1} dw \wedge d\overline{w}}{|w|^2}. \tag{2.7}$$

over p. Thus we get

$$\eta^{n+1} = \frac{(n+1)\lambda^n (f')^n f''}{|w|^2} (\pi^*\omega)^n \wedge \sqrt{-1} dw \wedge d\overline{w}. \tag{2.8}$$

Now observe that this expression of volume form is true not just over p. Indeed, if we choose a different trivialization $w' = q(z)w$, the expression (2.8) remains the same. So (2.8) holds everywhere on $U \times \mathbb{C}^*$.

Expression (2.7) indicates that, to make η positively definite, f should be a strictly convex function with $f' > 0$. So let us introduce

$$\tau = f'(s), \ \varphi(\tau) = f''(s). \tag{2.9}$$

Then, over p, the Ricci form is given by

$$\begin{aligned} \mathrm{Ric}\,(\eta) &= -\sqrt{-1}\partial\bar{\partial} \log \det(\eta) \\ &= \pi^*\mathrm{Ric}\,(\omega) - \left(n\lambda\frac{\varphi}{\tau} + \lambda\varphi'\right)\pi^*\omega \\ &\quad - \varphi\left(n\frac{\varphi}{\tau} + \varphi'\right)' \frac{\sqrt{-1} dw \wedge d\overline{w}}{|w|^2}. \end{aligned} \tag{2.10}$$

To build a metric η with special geometric features, it is natural to impose some conditions on the base metric ω. It turns out that there are several ways to do this.

(1) The most commonly used condition for ω is the Kähler–Einstein condition. Namely, we assume

$$\mathrm{Ric}\,(\omega) = \mu\omega \tag{2.11}$$

for some constant μ. In this setting, (2.10) becomes

$$\begin{aligned} \mathrm{Ric}\,(\eta) = &\left(\mu - n\lambda\frac{\varphi}{\tau} - \lambda\varphi'\right)\pi^*\omega \\ &- \varphi\left(n\frac{\varphi}{\tau} + \varphi'\right)'\frac{\sqrt{-1}dw\wedge d\overline{w}}{|w|^2}. \end{aligned} \tag{2.12}$$

From this we easily get the expression of the scalar curvature

$$S(\eta) = \frac{1}{\tau^n}\left(\frac{\mu\tau^{n+1}}{(n+1)\lambda} - \tau^n\varphi\right)'', \tag{2.13}$$

which now holds everywhere on $L^{-1}\backslash\{E_0\}$.

(2) A less common condition for ω is the Kähler–Einstein edge condition studied in [42], by which we mean

$$\mathrm{Ric}\,(\omega) = \mu\omega + 2\pi(1-\beta)[D] \tag{2.14}$$

for some constant μ, cone angle $\beta \in (0,1]$ and a smooth divisor D on X. We put

$$\overline{D} := \pi^* D.$$

In this case, the corresponding Hermitian metric h satisfying (2.1) is not supposed to smooth, but one can still derive (2.10) in the current sense:

$$\begin{aligned} \mathrm{Ric}\,(\eta) = &\left(\mu - n\lambda\frac{\varphi}{\tau} - \lambda\varphi'\right)\pi^*\omega + 2\pi(1-\beta)[\overline{D}] \\ &- \varphi\left(n\frac{\varphi}{\tau} + \varphi'\right)'\frac{\sqrt{-1}dw\wedge d\overline{w}}{|w|^2}. \end{aligned} \tag{2.15}$$

This allows one to construct Kähler–Einstein edge metrics, which will be explored in a forthcoming paper [46].

(3) One can also consider the twisted Kähler–Einstein condition. More precisely, we assume

$$\mathrm{Ric}\,(\omega) = \mu\omega + \alpha \tag{2.16}$$

for some constant μ and some non-negative $(1,1)$-form α on X, in which case, (2.10) reads

$$\begin{aligned} \mathrm{Ric}\,(\eta) = &\left(\mu - n\lambda\frac{\varphi}{\tau} - \lambda\varphi'\right)\pi^*\omega + \pi^*\alpha \\ &- \varphi\left(n\frac{\varphi}{\tau} + \varphi'\right)'\frac{\sqrt{-1}dw\wedge d\overline{w}}{|w|^2}. \end{aligned} \tag{2.17}$$

This case will be particularly useful for the proof of Theorem 1.3.

2.2. **Metric cone and normalized volume.**
In what follows we will additionally assume that L is *ample* (so that $\lambda > 0$). The goal of this part is review a standard construction of a family of metric cones over the circle bundle of X (see also [42]). The resulting metric cone will be denoted by (V, o^*), where o^* is the vertex.

Recall that when X is Fano and ω is Kähler–Einstein, a classical result of Kobayashi [25] gurantees the existence of Sasaki–Einstein metrics on certain circle bundles over X, which turns out can be constructed using Calabi unsatz (we refer the reader to the survey [50] for more information on this subject). For completeness we will include the details of this construction and then relate it to the comparison geometry and as a byproduct of these general discussions, we prove Theorem 1.3. See also [37] for a closely related discussion of this subject from an algebro-geometric viewpoint.

To construct a metric cone (V, o^*), we use the following observation. Geometrically we want E_0 to correspond to the vertex o^* of metric cone (namely, we want E_0 to shrink to a point). To make this happen, we go back to the local expression (2.7). As $|w| \searrow 0$, we want $\tau \searrow 0$, so that η degenerates along E_0. Moreover, to make the metric complete near the vertex (where $|w| \ll 1$), one should have

$$\lim_{\tau \to 0^+} \varphi(\tau) = 0. \tag{2.18}$$

On the other hand, it is tempting to make the Ricci form $\mathrm{Ric}\,(\eta)$ as simple as possible. So let us look at its expression (2.10). A natural candidate of φ that can simplify (2.10) is supposed to satisfy the following ODE:

$$n\frac{\varphi}{\tau} + \varphi' = \mathrm{Const.} \tag{2.19}$$

Using the boundary condition (2.18), we get

$$\varphi(\tau) = a\tau, \tag{2.20}$$

for some $a > 0$, in which case, (2.10) reduces to (see also [42, Lemma 1])

$$\mathrm{Ric}\,(\eta) = \pi^*\big(\mathrm{Ric}\,(\omega) - a(n+1)\lambda\omega\big), \tag{2.21}$$

so that $\mathrm{Ric}\,(\eta)$ is captured by the Ricci curvature of base manifold (X, ω) in a simple manner. Now using (2.9) and (2.20), we easily recover

$$f(s) = Ce^{as},$$

for some constant $C > 0$. By rescaling f, we may assume $C = \frac{1}{2}$ so that

$$f(s) = e^{as}/2 \tag{2.22}$$

and

$$\eta = \frac{\sqrt{-1}}{2}\partial\bar{\partial}e^{as}, \tag{2.23}$$

where $a > 0$ is a parameter.

In the following, we show that such η indeed gives rise to a metric cone (V, o^*). To see this, we introduce a new variable

$$r \in (0, \infty)$$

such that

$$r^2 = e^{as}. \tag{2.24}$$

Then r is also a globally defined function on $L^{-1}\backslash\{E_0\}$. In our chosen coordinate system, we have

$$r^2 = |w|^{2a}e^{a\lambda P(z)}. \tag{2.25}$$

We shall show the following

Proposition 2.1 ([5, 50])**.** *Let η be the Kähler form defined by (2.23). For any $a > 0$, the Riemannian metric g_η associated to η is a warped product:*

$$g_\eta = dr \otimes dr + r^2 g_{_M}, \tag{2.26}$$

where $g_{_M}$ is a Riemannian metric on the circle bundle $M := \{r = 1\}$ of X, so that $(L^{-1}\backslash\{E_0\}, g_\eta)$ indeed defines a metric cone.

Proof. We will prove this by explicitly calculating g_η in local coordinates (recall (2.4)). Indeed, over U we have

$$\begin{aligned}
\eta &= \frac{\sqrt{-1}}{2}\partial\bar{\partial}r^2 = \frac{\sqrt{-1}}{2}\partial\bar{\partial}\big(|w|^{2a}e^{a\lambda P(z)}\big)\\
&= \frac{\sqrt{-1}r^2}{2}\Bigg(a^2\frac{dw\wedge d\overline{w}}{|w|^2} + a\lambda\frac{\partial^2 P}{\partial z_i\partial\overline{z_j}}dz_i\wedge d\overline{z_j}\\
&\qquad + a^2\lambda^2\frac{\partial P}{\partial z_i}\frac{\partial P}{\partial\overline{z_j}}dz_i\wedge d\overline{z_j} + a^2\lambda\bigg(\frac{\partial P}{\partial z_i}dz_i\wedge\frac{d\overline{w}}{\overline{w}} + \frac{\partial P}{\partial\overline{z_j}}\frac{dw}{w}\wedge d\overline{z_j}\bigg)\Bigg)\\
&= \frac{a\lambda r^2}{2}\pi^*\omega + \frac{a^2r^2}{2}\frac{\sqrt{-1}dw\wedge d\overline{w}}{|w|^2} + \frac{\sqrt{-1}a^2\lambda^2r^2}{2}P_iP_{\overline{j}}dz_i\wedge d\overline{z_j}\\
&\qquad + \frac{\sqrt{-1}a^2\lambda r^2}{2}\bigg(P_idz_i\wedge\frac{d\overline{w}}{\overline{w}} + P_{\overline{j}}\frac{dw}{w}\wedge d\overline{z_j}\bigg).
\end{aligned} \tag{2.27}$$

Then the corresponding Riemannian metric tensor g_η is given by

$$\begin{aligned}
g_\eta = {}& \frac{a\lambda r^2}{2}\pi^*g_\omega + \frac{a^2r^2}{2}\frac{dw\otimes d\overline{w} + d\overline{w}\otimes dw}{|w|^2} + \frac{a^2\lambda^2r^2}{2}P_iP_{\overline{j}}\Big(dz_i\otimes d\overline{z_j} + d\overline{z_j}\otimes dz_i\Big)\\
&+ \frac{a^2\lambda r^2}{2}\bigg(P_i\frac{dz_i\otimes d\overline{w} + d\overline{w}\otimes dz_i}{\overline{w}} + P_{\overline{j}}\frac{d\overline{z_j}\otimes dw + dw\otimes d\overline{z_j}}{w}\bigg).
\end{aligned} \tag{2.28}$$

Now we write w using polar coordinates:

$$\begin{aligned}
w &= |w|e^{\sqrt{-1}\theta}\\
&= r^{1/a}e^{-\lambda P/2+\sqrt{-1}\theta},\ \theta\in[0,2\pi).
\end{aligned} \tag{2.29}$$

Here we used (2.25). From (2.29) we deduce

$$\begin{cases}
\frac{dw}{w} = \frac{1}{ar}dr - \frac{\lambda}{2}dP + \sqrt{-1}d\theta,\\
\frac{d\overline{w}}{\overline{w}} = \frac{1}{ar}dr - \frac{\lambda}{2}dP - \sqrt{-1}d\theta.
\end{cases} \tag{2.30}$$

Plugging (2.30) into (2.28), we obtain
(2.31)

$$\begin{aligned}
g_\eta =& \frac{a\lambda r^2}{2}\pi^* g_\omega + \frac{a^2 r^2}{2}\Bigg(\frac{2}{a^2r^2}dr\otimes dr - \frac{\lambda}{ar}(dr\otimes dP + dP\otimes dr) + \frac{\lambda^2}{2}dP\otimes dP + 2d\theta\otimes d\theta\Bigg)\\
&+ \frac{a^2\lambda^2 r^2}{2}P_i P_{\bar{j}}\Big(dz_i\otimes d\overline{z_j} + d\overline{z_j}\otimes dz_i\Big) + \frac{a^2\lambda r^2}{2}\Bigg(\frac{1}{ar}(dr\otimes dP + dP\otimes dr) - \lambda dP\otimes dP\\
&+ \sqrt{-1}\Big(P_{\bar{j}}(d\overline{z_j}\otimes d\theta + d\theta\otimes d\overline{z_j}) - P_i(dz_i\otimes d\theta + d\theta\otimes dz_i)\Big)\Bigg)\\
=& dr\otimes dr + r^2\Bigg[\frac{a\lambda}{2}\pi^* g_\omega + a^2 d\theta\otimes d\theta - \frac{a^2\lambda^2}{4}dP\otimes dP + \frac{a^2\lambda^2}{2}P_iP_{\bar{j}}\Big(dz_i\otimes d\overline{z_j} + d\overline{z_j}\otimes dz_i\Big)\\
&+ \frac{\sqrt{-1}a^2\lambda}{2}\Big(P_{\bar{j}}(d\overline{z_j}\otimes d\theta + d\theta\otimes d\overline{z_j}) - P_i(dz_i\otimes d\theta + d\theta\otimes dz_i)\Big)\Bigg]\\
=& dr\otimes dr + r^2\Bigg[\frac{a\lambda}{2}\pi^* g_\omega + a^2\Big(d\theta + \lambda d^{\mathbb{C}}P\Big)\otimes\Big(d\theta+\lambda d^{\mathbb{C}}P\Big)\Bigg],
\end{aligned}$$

where

$$d^{\mathbb{C}} := \frac{\sqrt{-1}}{2}(\overline{\partial} - \partial).$$

One can easily verify that, the real 1-form

$$\phi := d\theta + \lambda d^{\mathbb{C}}P \tag{2.32}$$

is globally defined on $L^{-1}\backslash\{E_0\}$. Indeed, given a different trivialization $\sigma' := q(z)\sigma$, where $q(z)$ is some locally defined nowhere vanishing holomorphic function on X, one has $w' = q^{-1}w$ and $P' = P + \frac{1}{\lambda}\log|q|^2$. The argument θ' of w' can be expressed by $\theta' = \theta + \frac{\sqrt{-1}}{2}(\log q - \log\overline{q})$. Then one has

$$\begin{aligned}
d\theta' + \lambda d^{\mathbb{C}}P' &= d\theta + \lambda d^{\mathbb{C}}P + \frac{\sqrt{-1}}{2}d(\log q - \log\overline{q}) + d^{\mathbb{C}}\log|q|^2\\
&= d\theta + \lambda d^{\mathbb{C}}P + \frac{\sqrt{-1}}{2}\left(\frac{\partial q}{q} - \frac{\overline{\partial q}}{\overline{q}} + \frac{\overline{\partial q}}{\overline{q}} - \frac{\partial q}{q}\right)\\
&= d\theta + \lambda d^{\mathbb{C}}P.
\end{aligned}$$

So (2.32) defines a global 1-form on $L^{-1}\backslash\{E_0\}$. Thus we obtain

$$g_\eta = dr\otimes dr + r^2\left(\frac{a\lambda}{2}\pi^*g_\omega + a^2\phi\otimes\phi\right). \tag{2.33}$$

Now let

$$M := \{r = 1\} \tag{2.34}$$

be the circle bundle over X and we define a 2-form

$$g_{_M} := \frac{a\lambda}{2}\pi^* g_\omega + a^2\phi\otimes\phi. \tag{2.35}$$

By (2.31), $g_{_M}$ can be locally expressed as

$$g_{_M} = \frac{a\lambda}{2}\pi^*g_\omega + a^2\Big(d\theta + \lambda d^{\mathbb{C}}P\Big)\otimes\Big(d\theta + \lambda d^{\mathbb{C}}P\Big) \tag{2.36}$$

This implies that $g_{_M}$ is a well defined symmetric 2-tensor when restricted to the circle bundle M since $(z_1, ..., z_n, \theta)$ forms a local coordinate system of M over $U \subset X$. Moreover $g_{_M}$ is clearly positively definite when restricted to M. Thus $g_{_M}$ induces a Riemannian metric on M, also denoted $g_{_M}$ by abuse of notation.

In summery, the Kähler metric g_η we deduced from the ODE (2.19) defines a cone metric on $L^{-1}\backslash\{E_0\}$, which degenerates E_0 to the vertex of the metric cone. Moreover g_η takes the form

$$g_\eta = dr \otimes dr + r^2 g_{_M},$$

where $g_{_M}$, given by (2.35), is a Riemannian metric defined on the circle bundle M of X. The proof is complete.

□

In particular, in the language of Sasakian geometry, $(M, g_{_M})$ is a regular Sasakian manifold. Note that in the work of Kobayashi [25], the 1-form ϕ is interpreted as a connection 1-form on M.

For any $a > 0$, the metric cone constructed above will be denoted by (V_a, o^*), where o^* the vertex. The manifold M is called the *link* of (V_a, o^*). Namely we have

$$V_a = C(M, g_{_M}) := (\mathbb{R}_+ \times M, dr^2 + r^2 g_{_M}) \cup \{o^*\}. \tag{2.37}$$

By the Gauss–Codazzi equations, one has

$$\mathrm{Ric}\,(g_\eta) = \mathrm{Ric}\,(g_{_M}) - 2n g_{_M}. \tag{2.38}$$

This relates the Ricci curvature of (V_a, o^*) to that of M.

As we shall see below, the geometry of (V_a, o^*) depends on the parameter a, since one can easily calculate the volume of the unit ball $B_1(o^*)$ of the metric cone.

Proposition 2.2 ([42])**.** *The volume of the unit ball $B_1(o^*)$ centered at the vertex of (V_a, o^*) is given by*

$$\mathrm{Vol}(B_1(o^*)) = \frac{(a\pi)^{n+1}\mathrm{Vol}(L)}{(n+1)!}. \tag{2.39}$$

Proof. We use η^{n+1} to compute the volume. Plugging (2.22) into (2.8) and using (2.5), we obtain

$$\eta^{n+1} = \frac{(n+1)e^{a\lambda(n+1)P}\lambda^n a^{n+2}}{2^{n+1}}(\pi^*\omega)^n \wedge \sqrt{-1}|w|^{2a(n+1)-2}dw \wedge d\overline{w}$$

Thus we get

$$\begin{aligned}
\mathrm{Vol}(B_1(o^*)) &= \int_{r\le 1} \frac{\eta^{n+1}}{(n+1)!} \\
&\overset{(2.25)}{=} \int_X \int_{|w|\le e^{-\lambda P/2}} \frac{(n+1)e^{a\lambda(n+1)P}\lambda^n a^{n+2}}{2^{n+1}(n+1)!}(\pi^*\omega)^n \wedge \sqrt{-1}|w|^{2a(n+1)-2}dw \wedge d\overline{w} \\
&= \int_X \frac{2\pi a^{n+1}\lambda^n}{2^{n+1}(n+1)!}\omega^n \\
&\overset{(2.1)}{=} \frac{(a\pi)^{n+1}\mathrm{Vol}(L)}{(n+1)!},
\end{aligned}$$

Here we used the fact that $\omega \in 2\pi c_1(L)/\lambda$.

□

From Proposition 2.2 we deduce the *nomalized volume* $\kappa(V_a, o^*)$ (the ratio between the cone volume and the Euclidean volume; see [43, (5.25)]) of the metric cone:

$$\kappa(V_a, o^*) := \frac{\mathrm{Vol}(B_1(o^*))}{\mathrm{Vol}(B_1(0^{2n+2}))} = a^{n+1}\mathrm{Vol}(L). \tag{2.40}$$

Here $\mathrm{Vol}(B_1(0^{2n+2})) = \pi^{n+1}/(n+1)!$ denotes the volume of the Euclidean unit $(2n+2)$-ball. In particular, the normalized volume $\kappa(V_a, o^*)$ only depends on $\mathrm{Vol}(L)$ and the parameter a.

Example 2.3.

(1) *The simplest example that fits into this framework is $X := \mathbb{P}^n$. We take $\omega := \omega_{FS} \in \mathcal{O}_{\mathbb{P}^n}(1)$. Let $L := \mathcal{O}_{\mathbb{P}^n}(1)$ and h^{-1} be the standard Hermitian metric on the tautological line bundle $L^{-1} = \mathcal{O}_{\mathbb{P}^n}(-1)$. Furthermore, we choose $a = 1$. In this case, $L^{-1}\backslash\{E_0\} \cong \mathbb{C}^{n+1}\backslash\{0\}$ and $M = S^{2n+1}$ (since in this case r is the distance function to the origin $0 \in \mathbb{C}^{n+1}$). Moreover $\eta = \sqrt{-1}\partial\bar{\partial}r^2/2$ is simply the standard flat Kähler form on $\mathbb{C}^{n+1}$ and $g_{_M}$ is the standard round metric on S^{2n+1}. One also has* $\mathrm{Vol}(L) = 1$. *Plugging this into (2.39), we obtain*

$$\mathrm{Vol}(B_1(o^*)) = \frac{\pi^{n+1}}{(n+1)!},$$

which is, of course, the volume of the unit $(2n+2)$-ball in the Euclidean space.

(2) *A more interesting example is the Stenzel cone [51]. We take*

$$X := \{X_0^2 + ... + X_{n+1}^2 = 0\} \subset \mathbb{P}^{n+1}$$

to be a smooth quadric in $\mathbb{P}^{n+1}$. Then X is a homogeneous Kähler–Einstein Fano manifold. Let $L := \mathcal{O}_X(1)$. Note that, by adjunction, one has $-K_X = nL$. We may choose $\omega \in 2\pi c_1(L)$ such that $\mathrm{Ric}\,(\omega) = n\omega$. *Choose a Hermitian metric h on L such that $R_h = \omega$. Now we pick $a = \frac{n}{n+1}$. Then (2.23) yeilds a Calabi–Yau cone (V, o^*), as in this case (2.21) gives* $\mathrm{Ric}\,(\eta) = 0$. *This cone metric η on $V\backslash\{o^*\}$ can be thought of as a Ricci flat metric defined on the germ of an ordinary double point:*

$$\{X_0^2 + ... + X_{n+1}^2 = 0\} \subset \mathbb{C}^{n+2}.$$

Note that $\mathrm{Vol}(L) = 2$. *Using (2.40), we obtain the normalized volume:*

$$\kappa(V, o^*) = 2\big(\frac{n}{n+1}\big)^{n+1}.$$

(3) *More generally, let X be a Fano manifold admitting a Kähler–Einstein metric $\omega \in 2\pi c_1(X)$ such that* $\mathrm{Ric}\,(\omega) = \omega$. *Let L be an ample line bundle on X equipped with a Hermitian metric h such that $R_h = \lambda\omega$ for some $\lambda > 0$. We choose $a = 1/((n+1)\lambda)$. Then (2.23) yeilds a Calabi–Yau cone (V, o^*) with* $\mathrm{Ric}\,(\eta) = 0$.

An interesting consequence of (2.40) is the the following result (see also [19, (2.22)]), which can be thought of as a weak version of the volume upper bound for Kähler–Einstein Fano varieties studied by Fujita [16] and Liu [41].

Proposition 2.4 ([19])**.** *There exists a dimensional constant $\epsilon(n) \in (0,1)$ such that the following holds. Let X be an n-dimensional Kähler–Einstein Fano manifold. Let $I(X)$ be its Fano index. Then one has*

$$\begin{cases} I(X)(-K_X)^n = (n+1)^{n+1}, & \text{when } X = \mathbb{P}^n \\ I(X)(-K_X)^n \leq \big(1-\epsilon(n)\big)(n+1)^{n+1}, & \text{otherwise.} \end{cases}$$

Proof. By definition of the Fano index, we may find an ample line bundle L such that

$$-K_X = I(X)L.$$

By the Kähler–Einstein condition, we may choose a Kähler form $\omega \in 2\pi c_1(L)$ such that

$$\mathrm{Ric}\,(\omega) = I(X)\omega.$$

Pick a Hermitian metric h on L such that

$$R_h = \omega.$$

Given these data, one can run the above ODE construction to obtain a family of metric cones (V_a, o^*). Now we choose

$$a = I(X)/(n+1).$$

Then (2.21) gives

$$\mathrm{Ric}\,(\eta) = 0,$$

i.e., (V, o^*) is a Calabi–Yau cone. So (2.38) implies that, the link M is a $(2n+1)$-dimensional Einstein manifold with Enstein constant $2n$

Now applying the Bishop–Gromov theorem to M, one has

$$\kappa(V_a, o^*) \leq 1,$$

and the equality holds if and only if (V_a, o^*) is *isometric* to the Euclidean space $\mathbb{C}^n$, in which case, X must be $\mathbb{P}^n$. Indeed, as we have seen in Example 2.3.1, $X = \mathbb{P}^n$ implies $\kappa(V_a, o^*) = 1$. Conversely, when $\kappa(V_a, o^*) = 1$, the S^1-bundle $(M, g_{_M})$ is isometric to the round sphere S^{2n+1}. Then one can compute the Reeb vector field, which generates the Hopf S^1-action on S^{2n+1}, so that the orbit space is simply $\mathbb{P}^{n+1}$.

Now let us turn to the case where $\kappa(V_a, o^*) < 1$. In this case the link M of the Calabi–Yau cone (V_a, o^*) is an $(2n+1)$-dimensional Einstein manifold with Enstein constant $2n$, but *not isometric* to the round sphere S^{2n+1}. Then the well-known sphere gap theorem (see e.g., [9, Theorem 4.3]) guarantees that there is an *Anderson constant* $\epsilon(n) \in (0,1)$ such that

$$\mathrm{Vol}(M, g_{_M}) \leq \big(1-\epsilon(n)\big)\mathrm{Vol}(S^{2n+1}, g_{_{S^{2n+1}}}),$$

so that

$$\kappa \leq 1 - \epsilon(n).$$

This completes the proof. ∎

Remark 2.5. *In fact the gap $\epsilon(n)$ can be specified; see Corollary 3.5.*

Similar strategy shows the following

Theorem 2.6 (=Theorem 1.3)**.** *Let L be an ample line bundle on a Fano manifold X. Then one has*

$$I(L)\beta(X,L)^{n+1}\mathrm{Vol}(L) \leq (n+1)^{n+1}.$$

Proof. As pointed out in Remark 1.4, by rescaling, we might as well assume

$$I(L) = 1.$$

Our goal is to prove

$$\beta(X, L)^{n+1}\mathrm{Vol}(L) \le (n+1)^{n+1}.$$

Given any $\mu \in (0, \beta(X, L))$, choose $\omega \in 2\pi c_1(L)$ such that

$$\mathrm{Ric}\,(\omega) \ge \mu\omega.$$

Pick a Hermitian metric on L with $R_h = \omega$. Then the ODE construction gives us a family of metric cones (V_a, o^*) depending on the parameter $a > 0$. Let us choose

$$a = \frac{\mu}{n+1},$$

in which case, (2.21) implies

$$\mathrm{Ric}\,(\eta) = \pi^*\big(\mathrm{Ric}\,(\omega) - \mu\omega\big) \ge 0.$$

So from (2.38) we deduce

$$\mathrm{Ric}\,(g_{_M}) \ge 2n g_{_M}.$$

Applying Bishop–Gromov theorem to M, one has (recall (2.40))

$$\kappa(V_a, o^*) = (\mu/n+1)^{n+1}\mathrm{Vol}(L) \le 1,$$

so that

$$\mu^{n+1}\mathrm{Vol}(L) \le (n+1)^{n+1}.$$

Letting $\mu \to \beta(X, L)$, we get

$$\beta(X, L)^{n+1}\mathrm{Vol}(L) \le (n+1)^{n+1},$$

as desired. □

As an immediate consequence, we have

Corollary 2.7 (=Corollary 1.5)**.** *Let L be an ample $\mathbb{Q}$-line bundle on a Fano manifold X, then one has*

$$I(L)\beta(X, L) \le n+1.$$

Proof. By rescaling, we assume

$$I(L) = 1.$$

In this case L is an integral ample line bundle, so

$$\mathrm{Vol}(L) = L^n \ge 1$$

is a positive integer. Then Theorem 1.3 implies that

$$\beta(X, L) \le n+1,$$

so that

$$I(L)\beta(X, L) \le n+1$$

holds for arbitrary ample $\mathbb{Q}$-line bundle L by scailing invariance. □

3. Proof of Theorem 1.6

In contrast to Section 2, the discussion in this part will be purely algebraic. The main goal is to prove Theorem 1.6.

3.1. **The δ-invariant.**
It turns out that the δ-invariant introduced recently in the literature is the right notion for us, which we now describe.

Following [17, 4], the δ-invariant of L is defined by

$$\delta(X,L) := \inf_{v\in \mathrm{Val}_X} \frac{A_X(v)}{S_L(v)}. \tag{3.1}$$

Here Val_X denotes the space of valuations over X, $A_X(v)$ denotes the log discrepancy of v, and $S_L(v)$ denotes the expected vanishing order of L with respect to v. Note that δ-invariant is also called *stability threshold* in the literature, which plays important roles in the study of K-stability and has attracted intensive research attentions. The following result proved in [3] (see also [11, Theorem 5.7]), which gives an geometric interpretation of the δ-invariants on Fano manifolds.

Theorem 3.1. *Let L be an ample line bundle on a Fano manifold X. Then one has*

$$\beta(X,L) = \min\{\epsilon(X,L), \delta(X,L)\}.$$

Regarding the δ-invariant, [4, Theorem D] implies the following volume upper bound :

$$\delta(X,L)^n \mathrm{Vol}(L) \leq (n+1)^n. \tag{3.2}$$

This inequality reveals the deep relationhip between *singularities* and *volumes* of linear systems.

With the help of δ-invariant, the statement of Theorem 1.3 can now be enhanced as follows.

We begin with the following result, which characterizes the equality of (1.8).

Proposition 3.2. *Let L be an ample $\mathbb{Q}$-line bundle on a Fano manifold X with*

$$I(L)\beta(X,L)^{n+1}\mathrm{Vol}(L) = (n+1)^{n+1},$$

then X is biholomorphic to $\mathbb{P}^n$.

Proof. By rescaling, we might as well assume [1]

$$\beta(X,L) = 1.$$

By Theorem 3.1, we deduce that

$$\delta(X,L) \geq 1.$$

Thus (3.2) implies

$$\mathrm{Vol}(L) \leq (n+1)^n.$$

On the other hand, by assumption, we have

$$I(L)\mathrm{Vol}(L) = (n+1)^{n+1}.$$

So we find that

$$I(L) \geq n+1.$$

Thus Corollary 1.5 forces that

$$I(L) = n+1,\ \mathrm{Vol}(L) = (n+1)^n \text{ and } \delta(X,L) = \beta(X,L) = 1.$$

[1] It is possible that $\beta(X,L)$ is irrational before rescaling, but this will not cause issues for our argument. In fact we believe that one always has $\beta(X,L) \in \mathbb{Q}$.

In particular, L is an integral ample line bundle with index $n+1$. We put

$$L = (n+1)H \text{ and } N := -K_X - L.$$

Then H is ample and N is nef (as $\epsilon(X, L) \geq \beta(X, L) = 1$).

We claim that N is a trivial line bundle. Suppose otherwise, then one has

$$H^0\big(K_X + (n+1)H\big) = H^0(-N) = 0.$$

Meanwhile, by Kodaira vanishing, we have

$$H^i(K_X + kH) = 0, \text{ for any } i \geq 1 \text{ and } k \geq 1.$$

Thus for $k \in \{1, 2, ..., n+1\}$, the Euler characteristic of $K_X + kH$ satisfies

$$\chi(K_X + kH) = h^0(K_X + kH) = h^0\big((k-n-1)H - N\big) = 0.$$

Then Riemann–Roch implies that $\chi(K_X + kH)$ is a degree n polynomial in k with $n+1$ roots, which forces that

$$\chi(K_X + kH) \equiv 0$$

for any integer k, contradicting the ampleness of H. So N is trivial as claimed.

Thus we have

$$L = -K_X,$$

so that the Fano index satisfies

$$I(X) = I(L) = n+1.$$

Then the criterion of Kobayashi–Ochiai [26] guarantees that X is biholomorphic to the complex projective space. □

Note that the above proof crucially used Theorem 3.1, (3.2) and the index $I(L)$. In fact, regarding the index $I(L)$, one has the following result, which refines Corollary 1.5.

Lemma 3.3 ([18, 23]). *Let L be an ample $\mathbb{Q}$-line bundle on a Fano manifold X, then one has*

$$\begin{cases} I(L)\epsilon(X,L) = n+1, & \text{when } X \cong \mathbb{P}^n; \\ I(L)\epsilon(X,L) = n, & \text{when } X \cong Q \subset \mathbb{P}^{n+1} \text{ is a smooth quadric;} \\ I(L)\epsilon(X,L) < n, & \text{otherwise.} \end{cases}$$

Proof. We sketh the proof for readers' convenience. By rescaling, we assume

$$\epsilon(X, L) = 1.$$

Let $L = I(L)H$ for some ample line bundle H. It was shown in [18, 23] that, if $K_X + nH$ is not nef, then $X \cong \mathbb{P}^n$. On the other hand, the assumption $\epsilon(X, L) = 1$ says that $-K_X - I(L)H$ is nef.

If $I(L) > n$, then $-K_X - nH$ is ample so $K_X + nH$ cannot be nef and hence $X \cong \mathbb{P}^n$, in which case $L = -K_X$ and $I(L) = n+1$.

If $I(L) = n$, then $-K_X - nH$ is nef. Meanwhile, in this case $X \neq \mathbb{P}^n$ so $K + nH$ is also nef. This implies that $-K_X - nH$ is numerically trivial and hence $-K_X \sim nH$ (numerical equivalence is the same as linear equivalence on Fano manifolds), so that [26] guarantees that X is a smooth quadric in $\mathbb{P}^{n+1}$.

□

Now we are ready to prove

Theorem 3.4 (=Theorem 1.6)**.** *Let L be an ample $\mathbb{Q}$-line bundle on a Fano manifold X, then one has*

$$\begin{cases} I(L)\beta(X,L)^{n+1}\mathrm{Vol}(L) = (n+1)^{n+1}, & \text{when } X \cong \mathbb{P}^n; \\ I(L)\beta(X,L)^{n+1}\mathrm{Vol}(L) = 2n^{n+1}, & \text{when } X \cong Q \subset \mathbb{P}^{n+1} \text{ is a smooth quadric;} \\ I(L)\beta(X,L)^{n+1}\mathrm{Vol}(L) < n(n+1)^n, & \text{otherwise.} \end{cases}$$

Proof. By rescaling we assume

$$\beta(X,L) = 1.$$

When $X \cong \mathbb{P}^n$ or $X \cong Q \subset \mathbb{P}^{n+1}$, we have $\epsilon(X,L) = \beta(X,L) = 1$ by the existence of KE metrics. Otherwise, one has $\epsilon(X,L) \geq 1$. So Lemma 3.3 implies

$$\begin{cases} I(X) = n+1, & \text{when } X \cong \mathbb{P}^n; \\ I(X) = n, & \text{when } X \cong Q \subset \mathbb{P}^{n+1} \text{ is a smooth quadric;} \\ I(X) < n, & \text{otherwise.} \end{cases}$$

On the other hand, Theorem 3.1 gives $\delta(X,L) \geq 1$, then (3.2) implies

$$\mathrm{Vol}(L) \leq (n+1)^n.$$

So the result follows. $\square$

As a simple consequence, one can specify the Anderson gap $\epsilon(n)$ appearing in Proposition 2.4.

Corollary 3.5. *One can choose the gap $\epsilon(n)$ in Proposition 2.4 to be*

$$\epsilon(n) = 1 - \frac{n}{n+1}$$

Note that this also follows easily from the volume upper bounds derived in [16].

Remark 3.6. *The ordinary double point conjecture* [48, Conjecture 1.2] *expects that one can even take*

$$\epsilon(n) = 1 - 2\left(\frac{n}{n+1}\right)^{n+1}.$$

4. Relation with the algebraic normalized volume

In this part we review the normalized volume introduced by Li [36] and then make some connections and remarks (see also [22, Appendix C]). To be in accordance with previous discussions, we will restrict ourselves to the smooth Fano setting. For general log $\mathbb{Q}$-Fano cases, we refer to [37].

Let X be an n-dimensional Fano manifold. Suppose that

$$L = -\lambda K_X \tag{4.1}$$

is an ample line bundle for some $\lambda \in \mathbb{Q}_+$. Put

$$R(X,L) := \bigoplus_{k\geq 0} H^0(X, kL) \tag{4.2}$$

and

$$V := \mathrm{Spec}\, R(X,L). \tag{4.3}$$

Then V is an affine variety obtained by contracting the zero section E_0 of the total space L^{-1}. In particular, V has an isolated singularity o^*, called the vertex of the affine cone V. See the figure below for an intuitive illustration.

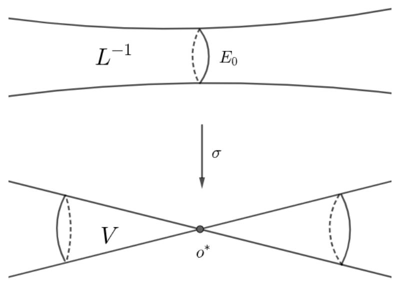

FIGURE 1. The affine cone (V, o^*)

Note that o^* is an isolated klt singularity of V. Indeed, we put

$$\widetilde{V} := L^{-1},$$

then $\widetilde{V} \xrightarrow{\epsilon} V$ resolves the singularity o^* of V and one can write

$$K_{\widetilde{V}} = \epsilon^* K_V + xE_0$$

for some $x \in \mathbb{Q}$. By adjunction, we get

$$-K_{E_0} = (x+1)\mathcal{O}_{\widetilde{V}}(E_0)|_{E_0}.$$

Using the fact that E_0 is a copy of X sitting inside $\widetilde{V}$ with normal bundle L^{-1}, one simply has (recall (4.1))

$$x = \frac{1}{\lambda} - 1 > -1,$$

so that

$$A_V(E_0) = \frac{1}{\lambda} > 0, \tag{4.4}$$

and hence o^* is a klt singularity. Here $A_V(\cdot)$ denotes the log discrepancy of divisors over V.

For the affine cone (V, o^*), Li [36] defined a purely algebraic notion called the *normalized volume*, denoted $\widehat{\mathrm{Vol}}(V, o^*)$, by

$$\widehat{\mathrm{Vol}}(V, o^*) := \inf_{v \in \mathrm{Val}_{V,o^*}} A_V(v)^{n+1} \cdot \mathrm{Vol}_{V,o^*}(v),$$

where Val_{V,o^*} denotes the space of valuations of V centered at o^*, $A_V(v)$ denotes the discrepancy of the valuation v and

$$\mathrm{Vol}_{V,o^*}(v) := \limsup_{m\to\infty} \frac{\mathrm{length}(\mathcal{O}_{V,o^*}/\mathfrak{a}_m(v))}{m^{n+1}/(n+1)!}$$

denotes the volume of v (see [37] for precise meanings of these notions).

The resolution $\widetilde{V} \xrightarrow{\epsilon} V$ gives a natural divisorial valuation $\mathrm{ord}_{E_0} \in \mathrm{Val}_{V,o^*}$, in which case, one has

$$A_V(\mathrm{ord}_{E_0}) = \frac{1}{a}$$

and

$$\mathrm{Vol}_{V,o^*}(\mathrm{ord}_{E_0}) = \limsup_{m\to\infty} \frac{h^0\Big(\widetilde{V}, \mathcal{O}_{\widetilde{V}}\big(-(m+1)E_0\big)\Big)}{m^{n+1}/(n+1)!}.$$

Using the exact sequence

$$0 \to \mathcal{O}_{\widetilde{V}}(-(m+1)E_0) \to \mathcal{O}_{\widetilde{V}}(-mE_0) \to \mathcal{O}_{E_0}(-mE_0) \to 0,$$

and $R^1\epsilon\big(\mathcal{O}_{\widetilde{V}}(-mE_0)\big) = 0$, one obtains

$$h^0\Big(\widetilde{V}, \mathcal{O}_{\widetilde{V}}\big(-mE_0\big)\Big) - h^0\Big(\widetilde{V}, \mathcal{O}_{\widetilde{V}}\big(-(m+1)E_0\big)\Big) = h^0\Big(E_0, -\mathcal{O}_{E_0}\big(-mE_0\big)\Big),$$

so that

$$\begin{aligned} \mathrm{Vol}_{V,o^*}(\mathrm{ord}_{E_0}) &= \limsup_{m\to\infty} \frac{\sum_{i=0}^m h^0(E_0, \mathcal{O}_{E_0}(-iE_0))}{m^{n+1}/(n+1)!} \\ &= \limsup_{m\to\infty} \frac{\sum_{i=0}^m h^0(X, i(-\lambda K_X))}{m^{n+1}/(n+1)!} \\ &= \limsup_{m\to\infty} \frac{n+1}{m} \cdot \frac{\sum_{i=0}^m h^0(E_0, \frac{i}{m}\cdot(-\lambda m K_X))}{m^n/n!} \\ &= (n+1)\int_0^1 \mathrm{Vol}\big(X, t(-\lambda K_X)\big)dt \\ &= (n+1)\int_0^1 (\lambda t)^n(-K_X)^n dt \\ &= \lambda^n(-K_X)^n. \end{aligned} \tag{4.5}$$

This gives

$$\widehat{\mathrm{Vol}}(V, o^*) \le \big(A_V(\mathrm{ord}_{E_0})\big)^{n+1} \cdot \mathrm{Vol}_{V,o^*}(\mathrm{ord}_{E_0}) = \frac{(-K_X)^n}{\lambda}. \tag{4.6}$$

Moreover, by [35, Theorem 1.1], when X is K-semistable (e.g., when X is Kähler–Einstein), (4.6) is in fact an equality.

Now assume that X is Kähler–Einstein, so that

$$\widehat{\mathrm{Vol}}(V, o^*) = \frac{(-K_X)^n}{\lambda}. \tag{4.7}$$

Recall that in this case (V, o^*) itself is also a Calabi–Yau cone with volume ratio (see Example 2.3.3)

$$\kappa(V, o^*) = \frac{L^n}{((n+1)\lambda)^{n+1}} = \frac{(-K_X)^n}{(n+1)^{n+1}\lambda}.$$

So one reads

$$\widehat{\mathrm{Vol}}(V, o^*) = (n+1)^{n+1}\kappa(V, o^*), \tag{4.8}$$

which relates the normalized volume of the affine cone (V, o^*) to its volume ratio as a Calabi–Yau metric cone. We refer the reader to [22, Appendix C] and [39, Corollary 5.7] for more general versions of this equality.

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