Peking Mathematical Journal
https://doi.org/10.1007/s42543-024-00093-9

ORIGINAL ARTICLE

Reduced Delta Invariant and Kähler–Einstein Metrics

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Received: 10 May 2024 / Revised: 16 September 2024 / Accepted: 27 October 2024

Abstract

Based on the pluripotential methods developed in Darvas and Zhang (Commun Pure Appl Math 77(12):4289–4327, 2024), we give a simplified prove for a result of Chi Li, which states that a log Fano vatiety admits a Kähler–Einstein metric if it has vanishing Futaki invariant and its reduced delta invariant is bigger than one.

Keywords Kähler–Einstein metric · K-stability · Yau–Tian–Donaldson conjecture

1 Introduction

Searching for canonical metrics on compact Kähler manifolds is a long-standing problem in the field of geometric analysis. Let X be a Fano manifold of dimension n. In this case, it is a difficult question to determine if X admits Kähler–Einstein (KE) metrics or not. By the solution of the Yau–Tian–Donadlson (YTD) conjecture, we now know that X admits a KE metric if and only if it is K-polystable; see, e.g., [21, 29] for proofs using Cheeger–Colding–Tian theory, [4, 12, 19, 22] for variational proofs and also [28, 32] for a proof using Tian's quantization methods.

In general, it is a highly non-trivial question to determine whether a given Fano manifold is K-polystable or not. Thanks to the Fujita–Li criterion [6, 13, 18] and the recent progress of Liu–Xu–Zhuang [24], one can now test K-polystability of a Fano manifold using the so-called reduced delta invariant that we now describe.

Let $\mathbb{G} := \mathrm{Aut}_0(X)$ denote the identity component of the biholomorphic automorphism group of X. If $\mathbb{G}$ is not reductive, then X cannot admit KE metrics by [26]. Therefore, in this paper, we always assume that $\mathbb{G}$ is a reductive group. Let $\mathbb{T}$ be a maximal torus of $\mathbb{G}$. By Lie group theory, $\mathbb{T}$ is non-trivial if $G \neq \{1\}$. Assume that $\mathbb{T} = (\mathbb{C}^*)^r$, $\mathbb{T}_{\mathbb{R}} := (S^1)^r$, $N_{\mathbb{Z}} := \mathrm{Hom}(\mathbb{C}^*, \mathbb{T})$, $N_{\mathbb{Q}} := N_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$, and $N_{\mathbb{R}} := N_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$.

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We first recall the usual delta invariant in the K-stability theory, going back to [6, 14]. It is defined as

$$\delta(X) := \inf_{v \in X^{\mathrm{div}}} \frac{A_X(v)}{S_X(v)},$$

where the inf is over all divisorial valuations v on X, namely, those valuations of the form $v = c\,\mathrm{ord}_F$, where $c \in \mathbb{Q}_{>0}$ and F is a prime divisor on some birational model $Y \xrightarrow{\pi} X$. Here

$$A_X(v) := c\left(1 + \mathrm{ord}_F(K_Y - \pi^* K_X)\right)$$

is the log discrepancy of v. And

$$S_X(v) := \frac{c}{\mathrm{vol}(-K_X)} \int_0^\infty \mathrm{vol}(\pi^*(-K_X) - xF)\mathrm{d}x$$

is called the expected vanishing order of $-K_X$ with respect to v. The significance of the invariant $\delta(X)$ lies in the fact that X is K-stable if and only if $\delta(X) > 1$, as shown in the recent work of Liu–Xu–Zhuang [24]. However, to detect the more subtle K-polystability, one needs to modify the definition of delta invariant by taking the $\mathbb{T}$-action on X into account. This leads to the notion of reduced delta invariant, as we now turn to describe.

The reduced delta invariant of X is defined by (following [19, Theorem 1.3] and [31, Appendix A]):

$$\delta^r(X) := \inf_{v \in X^{\mathrm{div}}_{\mathbb{T}}} \sup_{\xi \in N_{\mathbb{Q}}} \frac{A_X(v^\xi)}{S_X(v^\xi)},$$

where the inf is now over all $\mathbb{T}$-invariant divisorial valuations v on X, and v^ξ is the twist of v by $\xi \in N_{\mathbb{Q}}$ (which can be viewed as some $\mathbb{C}^*$-action on X). See (8) for a more precise definition of this twist. It is possible that $v^\xi = 0$ for some ξ (i.e., v^ξ is a trivial valuation; this happens for instance when X is toric), in which case we use the convention that $\frac{A_X(v^\xi)}{S_X(v^\xi)} = +\infty$. If v^ξ is non-trivial, then it is still a $\mathbb{T}$-invariant divisorial valuation, in which case $A_X(v^\xi)$ is the log discrepancy and $S_X(v^\xi)$ is the expected vanishing order as defined above. If $\mathbb{G} = \{1\}$, then one simply has $\delta^r(X) = \delta(X)$, reducing to the usual delta invariant.

In this work, we give a short proof of the following result going back to Li [19].

Theorem 1.1 *Let X be a Fano manifold. Then, X admits a KE metric if X has vanishing Futaki invariant and $\delta^r(X) > 1$.*

See Theorem 4.1 for the more general case of log Fano varieties. Note that by the recent work of Liu–Xu–Zhuang [24, Theorem 1.6], X is K-polystable if and only if X has vanishing Futaki invariant and $\delta^r(X) > 1$. Therefore, modulo [24] (together with

[2]), we get a simplified variational proof of the Yau–Tian–Donaldson conjecture for log Fano varieties.

Compared to Li's work, our proof relies on the pluripotential techniques recently developed in [12], which does not involve the use of K-energy or test configurations. We will show that the Ding functional is proper modulo $\mathbb{T}$ in a more direct way. Another novelty of our proof is that it is insensitive to singularities, hence works equally well for more general Kähler–Einstein metrics on log Fano varieties. This also allows us to avoid the non-trivial perturbation trick used in [19, 21, 22], producing another simplification for the proof of YTD conjectures in the singular setting. To stay focused, we will first treat the case when X is smooth and finally show in §4 how our techniques easily adapt to the singular setting.

However, we should emphasize that the KE metric produced by variational approach does not tell much information about its behavior around singularity, e.g., finite diameter and conical structure, so the metric geometry approach in [21, 29] still has advantages, which actually has played crucially roles in the construction of the K-moduli space (cf. [23, 31]).

Compared to the YTD type results proved in our previous work [12], the existence of non-trivial automorphisms will make the argument more involved. Indeed, when $\mathbb{G}$ is trivial, based on the analysis from [12], one can show the existence of Kähler–Einstein metrics from $\delta(X) > 1$ almost immediately (see Theorem 3.1 for a quick proof). However, to deal with automorphisms, we need to exploit the birational geometry of the product space $X \times \mathbb{C}$ as in [4, 16, 19]. To be more precise, we need to show that the crucial non-Archimedean estimate [19, Lemma 3.5] for test configurations actually holds for any finite energy sublinear subgeodesic rays as well. To show the geodesic stability of Ding functional, we also need to sharpen the Lelong number estimate in [12, Proposition 4.5]. Moreover, in the presence of the $\mathbb{T}$-action, to construct a non-trivial destabilizing ray as in [12, Theorem 5.3], we need some additional careful estimates for the J-functional, which borrows some ideas from [20, Proposition 6.2].

Further Directions. To stay brief, we do not treat the following interesting related directions in this paper. First, following [16], it is possible to further extend the scope of our results to the case of twisted soliton metrics on Fano type varieties after replacing E and S_L with their g-weighted versions. Also, our approach seems adaptable to the case of KE metrics with prescribed singularity type, as recently studied in [30]. Finally, it is desirable to extend our treatment to the cscK problem on general algebraic manifolds.

2 Preliminaries

2.1 Subgeodesic Rays and Test Curves

Let X be a Fano manifold, as in the introduction. Fix a $\mathbb{T}_{\mathbb{R}}$-invariant Kähler form $\omega \in c_1(X)$. Denote by PSH_ω the set of ω-plurisubharmonic (psh) functions on X, $\mathcal{E}^1_\omega$ the set of finite energy ω-psh functions, and let $\mathcal{E}^{1,\mathbb{T}_{\mathbb{R}}}_\omega$ be the set of $\mathbb{T}_{\mathbb{R}}$-invariant elements in $\mathcal{E}^1_\omega$. For any sublinear subgeodesic ray $\{u_t\} \subset \mathcal{E}^1_\omega$ (which will always be assumed to emanate from $0 \in \mathcal{E}^1_\omega$), let $\{\hat{u}_\tau\} \subset \mathrm{PSH}_\omega$ denote the test curve associated with $\{u_t\}$, which first appeared in the work of Ross–Witt Nyström [27, §5]. More precisely, $\{\hat{u}_\tau\}$ is given by

$$\hat{u}_\tau(x) := \inf_{t>0}\{u_t(x) - t\tau\}, \quad \tau \in \mathbb{R},\ x \in X.$$

That $\{u_t\}$ is sublinear implies the existence of a finite number, denoted by $\tau^+_{\hat{u}}$, such that

$$\tau^+_{\hat{u}} = \inf\{\tau \in \mathbb{R} : \hat{u}_\tau \equiv -\infty\}.$$

By the convexity of $t \mapsto u_t$, the map $t \mapsto \frac{\sup u_t}{t}$ is non-decreasing. Put $a := \lim_{t\to\infty} \frac{\sup u_t}{t}$. From the definition of $\hat{u}_\tau$, it is clear that $\tau^+_{\hat{u}} \leq a$. On the other hand, for any $\tau < a$, there exists $x \in X$, such that $u_t(x) \geq t\tau$ for all $t \gg 1$. Using the convexity of $t \mapsto u_t(x) - t\tau$, it follows that $\hat{u}_\tau(x) > -\infty$, so $a \leq \tau^+_{\hat{u}}$. Thus

$$\lim_{t\to\infty} \frac{\sup u_t}{t} = \tau^+_{\hat{u}}.$$

Note that these relations also hold for any finite energy sublinear subgeodesic rays in transcendental big classes on compact Kähler manifolds. We refer the reader to [12, §3] for a comprehensive treatment of this subject.

2.2 Ding Functional and Properness

Fix $h \in C^\infty(X, \mathbb{R})$, such that

$$\mathrm{Ric}(\omega) = \omega + \mathrm{dd^c} h,$$

with $\mathrm{dd^c} := \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}$. Set

$$V := \int_X \omega^n = \mathrm{vol}(-K_X).$$

For any $u \in \mathcal{E}^1_\omega$, put

$$L(u) := -\log \int_X e^{h-u}\omega^n,$$

and let

$$E(u) := \frac{1}{(n+1)V}\int_X u \sum_{i=0}^{n} \omega^i \wedge \omega_u^{n-i}$$

be the Monge–Ampère energy. Let

$$D(u) = L(u) - E(u), \quad u \in \mathcal{E}^1_\omega$$

denote the Ding functional.

Another function that is important in this work is the J-functional

$$J(u) := J(\omega, \omega_u) := \frac{1}{V}\int_X u\omega^n - E(u).$$

We put

$$J_{\mathbb{T}}(u) := \inf_{\sigma\in\mathbb{T}} J(\omega, \sigma^*\omega_u).$$

Definition 2.1 The Ding functional D is said to be proper modulo $\mathbb{T}$ if there exist $\varepsilon, C > 0$, such that

$$D(u) \geq \varepsilon J_{\mathbb{T}}(u) - C, \quad u \in \mathcal{E}^{1,\mathbb{T}_{\mathbb{R}}}.$$

By the variational principle [9] (and also [17]), the above properness of Ding functional is equivalent to the existence of KE metrics.

It should be well known to experts that J-functional is almost linear along geodesics. We record this fact in the next lemma, which will be needed in the proof of Theorem 1.1.

Lemma 2.2 *There exists C_0 depending only on (X, ω), such that the following holds. Let $[0, T] \ni t \mapsto u_t \in \mathcal{E}^1$ be any finite energy geodesic segment joining 0 and u_T, then for any $t \in [0, T]$, one has*

$$\frac{t}{T}J(u_T) - C_0 \leq J(u_t) \leq \frac{t}{T}J(u_T) + C_0.$$

Proof It is well known that there exists C_0, such that (see, e.g., [8, Lemma 3.45])

$$\sup u - C_0 \leq \frac{1}{V}\int_X u\omega^n \leq \sup u, \quad u \in \mathcal{E}^1_\omega.$$

Therefore

$$\sup u - E(u) - C_0 \leq J(u) \leq \sup u - E(u). \tag{1}$$

Moreover, it is well known that $t \mapsto \sup u_t - E(u_t)$ is linear. Therefore, we conclude. □

For any energy functional F defined on $\mathcal{E}^1_\omega$ and any sublinear subgeodesic ray $\{u_t\}_t \subset \mathcal{E}^1_\omega$, define the radial functional

$$F\{u_t\} := \liminf_{t\to\infty} \frac{F(u_t)}{t}.$$

When $F \in \{E, J\}$, the liminf is actually a limit. One has the following radial formulas (see [27, (6.2)], [10, (3.9)] and [12, (8), (31)]):

$$L\{u_t\} = \sup\left\{\tau : \int_X e^{-\hat{u}_\tau}\omega^n < \infty\right\}, \tag{2}$$

$$E\{u_t\} = \frac{1}{V}\int_{-\infty}^{\tau_{\hat{u}}^+}\left(\int_X \omega_{\hat{u}_\tau}^n - \int_X \omega^n\right)\mathrm{d}\tau + \tau_{\hat{u}}^+, \tag{3}$$

$$J\{u_t\} = \tau_{\hat{u}}^+ - E\{u_t\}. \tag{4}$$

Note that (4) is a consequence of (1) and that $\lim_{t\to\infty}\frac{\sup u_t}{t} = \tau_{\hat{u}}^+$. We say $\{u_t\}$ is non-trivial if $J\{u_t\} > 0$, i.e., $\tau_{\hat{u}}^+ > E\{u_t\}$.

2.3 Divisorial Valuations and Gauss Extension

Let X^{div} denote the space of non-trivial divisorial valuations on X, namely, the valuations of the form $c\,\mathrm{ord}_F$, where $c \in \mathbb{Q}_{>0}$ and $F \subset Y \xrightarrow{\pi} X$ is a prime divisor over X. Let

$$A_X(\mathrm{ord}_F) := A_X(F) := 1 + \mathrm{ord}_F(K_Y - \pi^* K_X)$$

be the log discrepancy of ord_F with respect to X. More generally, for $v = c\,\mathrm{ord}_F$

$$A_X(v) := cA_X(\mathrm{ord}_F).$$

If $v = 0$ is a trivial valuation, then we put $A_X(v) = 0$. Given any model $W \xrightarrow{\mu} X$ over X and $v \in X^{\mathrm{div}}$, one can find $\tilde{v} \in W^{\mathrm{div}}$ satisfying

$$\mu_*\tilde{v} = v.$$

Here, the push forward $\mu_*\tilde{v}$ is the valuation on X, such that

$$\mu_*\tilde{v}(f) = \tilde{v}(f \circ \mu) \quad \text{for any } f \in K(X).$$

Then, one has

$$A_X(v) = A_W(\tilde{v}) + \tilde{v}(K_W - \mu^* K_X). \tag{5}$$

In the literature, people usually write $\tilde{v} = v$ by abuse of notation, when the birational map μ is chosen without ambiguity. However, in our discussion below [especially when proving (10) and (17)], we have to be more precise, since there will be two different μ's in question.

Following [7, §4.1], we consider

$$X_{\mathbb{C}} := X \times \mathbb{C}.$$

Then, the function fields satisfy $K(X_{\mathbb{C}}) = K(X)(z)$, where z denotes the standard coordinate on $\mathbb{C}$. For any $f \in K(X)[z]$ in the Laurent polynomial ring, assume that

$$f = \sum_{\lambda \in \mathbb{N}} f_\lambda z^\lambda.$$

The Gauss extension $G(v)$ of $v \in X^{\mathrm{div}}$ is the unique $\mathbb{C}^*$-invariant extension of v, such that $G(v)(z) = 1$. Its value on f is explicitly given by

$$G(v)(f) := \min_{\lambda \in \mathbb{N}} \{v(f_\lambda) + \lambda\},$$

and for any $h = f/g \in K(X)(z)$ with $f, g \in K(X)[z]$, one has

$$G(v)(h) := G(v)(f) - G(v)(g).$$

If $v = 0$ is trivial, then one simply has $G(v) = \mathrm{ord}_{X \times \{0\}}$, which is a divisorial valuation on $X_{\mathbb{C}}$.

A fact we shall need is that $G(v)$ is also a divisorial valuation on $X_{\mathbb{C}}$ for any $v \in X^{\mathrm{div}}$. More precisely, assume that $v = c\,\mathrm{ord}_F$, with $c = \frac{a}{b}$, where a, b are two coprime positive integers. Then, $G(v)$ is of the form $\frac{1}{b}\mathrm{ord}_E$ for some prime divisor E over $X_{\mathbb{C}}$. One can explicitly describe E as follows. Let $X' \xrightarrow{\pi} X$ be a modification, such that $F \subset X'$ is a prime divisor. Consider $X'_{\mathbb{C}} := X' \times \mathbb{C}$ and $F_{\mathbb{C}} := F \times \mathbb{C}$. Then, E is a prime divisor over $X'_{\mathbb{C}}$, such that

$$\mathrm{ord}_E(X'_0) = b, \quad \mathrm{ord}_E(F_{\mathbb{C}}) = a.$$

And the log discrepancy $A_X(F)$ and $A_{X_{\mathbb{C}}}(E)$ are related by (see [7, Proposition 4.11])

$$A_{X_{\mathbb{C}}}(E) = b + aA_X(F),$$

so that

$$A_{X_{\mathbb{C}}}(G(v)) = 1 + A_X(v). \tag{6}$$

Moreover, E is $\mathbb{C}^*$-invariant so that its center on $X_{\mathbb{C}}$ is contained in $X_0 := X \times \{0\}$. In particular, if Ψ is a quasi plurisubharmonic (qpsh) function defined in a neighborhood of X_0, one can make sense of $G(v)(\Phi)$ by

$$G(v)(\Phi) := \frac{1}{b}\nu(\Phi, E),$$

where $\nu(\Phi, E)$ is the Lelong number of Φ along E (see [12, (13)] for our convention for the Lelong number).

We recall the following result [11, Proposition 3.1] (see also [4, §4.3]).

Lemma 2.3 *Let $\{u_t\}_t \subset \mathcal{E}^1(X, \omega)$ be a subgeodesic ray with $u_t \le 0$ for $t \ge 0$. Let U denote the qpsh function on $X \times \Delta^*$ given by*

$$U(x, z) := u_{-\log|z|^2}(x).$$

Note that U extends to a qpsh function on $X \times \Delta$. One has

$$\sup_\tau\{\tau - v(\hat{u}_\tau)\} = -G(v)(U)$$

for any $v \in X^{\mathrm{div}}$. If $v = 0$, then $-G(v)(U) = \tau^+_{\hat{u}}$.

Recall that, for $v = \lambda \mathrm{ord}_F \in X^{\mathrm{div}}$ and $\phi \in \mathrm{PSH}_\omega$, the notation $v(\phi)$ appearing above simply means that

$$v(\phi) := \lambda\nu(\phi, F).$$

As a consequence, we have the following.

Proposition 2.4 *For any subgeodesic ray $\{u_t\}_t \subset \mathcal{E}^{1,\mathbb{T}_\mathbb{R}}_\omega$ with $u_t \le 0$, one has*

$$L\{u_t\} = \inf_{v \in X^{\mathrm{div}}_{\mathbb{T}}}\{A_X(v) - G(v)(U)\}.$$

Proof One has (cf. [11, Theorem 5.7] and [4, (3.2)])

$$L\{u_t\} = \inf_{v \in X^{\mathrm{div}}_{\mathbb{T}}} \sup_{\tau \in \mathbb{R}}\{A_X(v) + \tau - v(\hat{u}_\tau)\}. \tag{7}$$

In [11] this identity is proved without involving $\mathbb{T}$-action. The same proof works for the equivariant setting. Thus, our assertion follows from the previous lemma. □

We remark that (7) also holds for general big classes, as considered in [11].

2.4 Valuations on *T*-Varieties

Following [19, §2.4], let $\mathbb{T}$ be an algebraic torus acting faithfully on X. By the structure theory of T-varieties, X is birationally a torus fibration over the Chow quotient of X by $\mathbb{T}$, which will be denoted by $Z := X//\mathbb{T}$. As a consequence, the function field $K(X)$ is the quotient field of the Laurent polynomial algebra

$$K(Z)[M_\mathbb{Z}] = \bigoplus_{\alpha \in M_\mathbb{Z}} K(Z) \cdot 1^\alpha,$$

where $M_\mathbb{Z} := \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*)$. Our convention (as in [19]) for the $\mathbb{T}$-action on $K(X)$ is that

$$\mathbf{t} \cdot f := f \circ \mathbf{t}^{-1}, \quad f \in K(X),\ \mathbf{t} \in \mathbb{T}.$$

And we put

$$K(X)_\alpha := \{f \in K(X) \,|\, \mathbf{t} \cdot f = \mathbf{t}^\alpha f\},\ \alpha \in M_{\mathbb{Z}}.$$

We say a valuation v on X is $\mathbb{T}$-invariant if $v(\mathbf{t} \cdot f) = v(f)$ for any $f \in K(X)$ and $\mathbf{t} \in \mathbb{T}$. By [1, p. 236] (cf. also [7, Lemma 4.2]), any $\mathbb{T}$-invariant valuation is of the form $v_{\mu,\zeta}$, where μ is a valuation on Z and $\zeta \in N_{\mathbb{R}}$, and for $f = \sum_\alpha f_\alpha 1^\alpha \in K(X)$, one has

$$v_{\mu,\zeta}(f) = \min_\alpha \{\mu(f_\alpha) + \langle \alpha, \zeta \rangle\}.$$

Given a $\mathbb{T}$-invariant valuation $v = v_{\mu,\zeta}$ and $\xi \in N_{\mathbb{R}}$, let

$$v^\xi := v_{\mu,\zeta+\xi} \tag{8}$$

be the twist of v by ξ, which is still a $\mathbb{T}$-invariant valuation. If v is a $\mathbb{T}$-invariant divisorial valuation and $\xi \in N_{\mathbb{Q}}$, then v^ξ is also divisorial if it is non-trivial (as v^ξ is Abhyankar and its rational rank of v^ξ is one; see [7, §1.3]). We denote by

$$X_{\mathbb{T}}^{\mathrm{div}}$$

the set of $\mathbb{T}$-invariant divisorial valuation on X.

Now, assume that $\xi \in N_{\mathbb{Z}}$. Then, it induces an automorphism η_ξ of $X \times \mathbb{C}^*$ as follows:

$$\eta_\xi \cdot (x, z) := (\xi(z) \cdot x, z), \quad (x, z) \in X \times \mathbb{C}^*,$$

which yields a birational map from $X_{\mathbb{C}}$ to itself. This further induces an action on the function field $K(X_{\mathbb{C}})$. More precisely, for any $h \in K(X_{\mathbb{C}})$, one has

$$\eta_\xi \cdot h := h \circ \eta_\xi^{-1} = h \circ \eta_{-\xi}.$$

Therefore, in particular, if $f \in K(X)_\alpha$, one has

$$\eta_\xi \cdot \bar{f} = \sum_{\alpha \in M_{\mathbb{Z}}} z^{\langle \alpha, \xi \rangle} \bar{f},$$

where $\bar{f}$ is the pull back of f on $X_{\mathbb{C}}$. Then, for any $v \in X_{\mathbb{T}}^{\mathrm{div}}$, this implies that

$$G(v)(\bar{f} \circ \eta_{-\xi}) = G(v^\xi)(\bar{f}), \quad f \in K(X)_\alpha,\ \alpha \in M_{\mathbb{Z}},$$

which further implies that (using the $\mathbb{T} \times \mathbb{C}^*$-invariance of $(\eta_{-\xi})_* G(v)$ and $G(v^\xi)$)

$$(\eta_{-\xi})_* G(v) = G(v^\xi). \tag{9}$$

Now, following the proof of [19, Proposition 3.3], let $\mathcal{W}$ be a birational model resolving the map $\eta_{-\xi}$:

$$\begin{array}{ccc} & \mathcal{W} & \\ {}^{\mu_1}\swarrow & & \searrow^{\mu_2} \\ X_{\mathbb{C}} & \overset{\eta_{-\xi}}{\dashrightarrow} & X_{\mathbb{C}} \end{array}$$

Let $\mathcal{V}$ be the divisorial valuation on $\mathcal{W}$, such that

$$(\mu_1)_*\mathcal{V} = G(v).$$

From the above commutative diagram, we have that

$$(\eta_{-\xi})_* G(v) = (\mu_2)_*\mathcal{V}.$$

Then, one can write (cf. (5))

$$\begin{aligned} A_{X_{\mathbb{C}}}(G(v)) &= A_{\mathcal{W}}(\mathcal{V}) + \mathcal{V}(K_{\mathcal{W}} - \mu_1^* K_{X_{\mathbb{C}}}), \\ A_{X_{\mathbb{C}}}((\eta_{-\xi})_* G(v)) &= A_{\mathcal{W}}(\mathcal{V}) + \mathcal{V}(K_{\mathcal{W}} - \mu_2^* K_{X_{\mathbb{C}}}). \end{aligned}$$

Therefore, from (9), we infer that

$$A_{X_{\mathbb{C}}}(G(v^{\xi})) - A_{X_{\mathbb{C}}}(G(v)) = \mathcal{V}(\mu_1^* K_{X_{\mathbb{C}}} - \mu_2^* K_{X_{\mathbb{C}}}).$$

Using (6), we further deduce that

$$A_X(v^{\xi}) - A_X(v) = \mathcal{V}(\mu_1^* K_{X_{\mathbb{C}}} - \mu_2^* K_{X_{\mathbb{C}}}). \tag{10}$$

Note that the above argument works for general log pairs as well.

2.5 Twisting Subgeodesic Rays

Let $\xi \in N_{\mathbb{Z}} = \mathrm{Hom}(\mathbb{C}^*, \mathbb{T})$. It determines a $\mathbb{C}^*$-action on X, namely, $\xi : \mathbb{C}^* \ni z \mapsto (z^{\xi_1}, ..., z^{\xi_r}) \in \mathbb{T}$. This, in particular, gives rise to a holomorphic vector field, say $V_\xi \in H^0(X, T_X^{1,0})$. The map $\xi \mapsto V_\xi$ can be extended $\mathbb{R}$-linearly to $N_{\mathbb{R}}$; thus, any $\xi \in N_{\mathbb{R}}$ can be identified with a holomorphic vector field V_ξ on X. We will be interested in the real part $\mathrm{Re}V_\xi$ of V_ξ, which generates a one-parameter subgroup

$$\{\sigma_\xi(t)\}_{t\in\mathbb{R}}.$$

The next example clarifies the notation chosen above.

Example 2.5 Assume that $\mathbb{T} = (\mathbb{C}^*)^n$ acts on $\mathbb{P}^n$ by

$$(z_1, ..., z_n) \cdot [W_0, W_1, ..., W_n] := [W_0, z_1 W_1, ..., z_n W_n].$$

Then, for any $\xi = (\xi_1, ..., \xi_n) \in N_{\mathbb{Z}}$ and $z \in \mathbb{C}^*$, $\xi(z)$ acts on $\mathbb{P}^n$ by

$$\xi(z) \cdot [W_0, W_1, ..., W_n] := [W_0, z^{\xi_1} W_1, ..., z^{\xi_n} W_n].$$

Moreover, for any $\xi = (\xi_1, ..., \xi_n) \in N_{\mathbb{R}}$, the vector field V_ξ can be expressed on the affine chart $\{W_0 \neq 0\}$ as

$$V_\xi = \sum_{i=1}^{n} \xi_i \left(x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i} \right) - \sqrt{-1} \sum_{i=1}^{n} \xi_i \left(x_i \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial x_i} \right),$$

where $x_i + \sqrt{-1} y_i = W_i / W_0$ are the coordinates on $\{W_0 \neq 0\}$. And the one-parameter subgroup σ_ξ acts on $\mathbb{P}^n$ by

$$\sigma_\xi(t) \cdot [W_0, W_1, ..., W_n] := [W_0, e^{\xi_1 t} W_1, ..., e^{\xi_n t} W_n], \quad t \in \mathbb{R}.$$

Now, we recap a standard construction going back to Mabuchi [25, §5]. Given $\xi \in N_{\mathbb{R}}$, one can define a smooth geodesic ray in $\mathcal{H}_\omega$.

Recall that $h \in C^\infty(X, \mathbb{R})$ is chosen, such that $\mathrm{Ric}(\omega) = \omega + \mathrm{dd^c} h$. Then, put

$$\psi_t^\xi := -\log \frac{\sigma_\xi(t/2)^*(e^h \omega^n)}{e^h \omega^n}, \quad t \geq 0. \tag{11}$$

One can easily check that $\psi_t^\xi \in \mathcal{H}_\omega$ and

$$\omega + \mathrm{dd^c} \psi_t^\xi = \sigma_\xi \left(\frac{t}{2} \right)^* \omega.$$

It is well known that $\{\psi_t^\xi\}$ thus defined is a geodesic ray. Moreover, direct calculation shows that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} E(\psi_t^\xi) &= \frac{1}{V} \int_X \dot{\psi}_t^\xi \left(\sigma_\xi \left(\frac{t}{2} \right)^* \omega^n \right) \\ &= \frac{-1}{V} \int_X \frac{\mathrm{d}}{\mathrm{d}t} \left(\sigma_\xi \left(\frac{t}{2} \right)^* h \right) (\sigma_\xi(t)^* \omega^n) \\ &= \frac{-1}{2V} \int_X \mathrm{Re}\, V_\xi(h) \omega^n = \frac{1}{2} \mathrm{Fut}(\mathrm{Re}\, V_\xi). \end{aligned}$$

Here, $\mathrm{Fut}(\cdot)$ denotes the Futaki invariant [15].

The above construction can be extended as follows. For any subgeodesic segment $\{u_t\} \subset \mathcal{E}_\omega^1$ with $t \in (a, b)$, $0 \leq a < b \leq \infty$, one can twist the segment using ξ by putting

$$u_t^\xi := -\log \frac{\sigma_\xi(t/2)^*(e^{h - u_t} \omega^n)}{e^h \omega^n} = \sigma_\xi \left(\frac{t}{2} \right)^* u_t + \psi_t^\xi, \quad t \in (a, b). \tag{12}$$

Then, $u_t^\xi \in \mathcal{E}^1_\omega$ and

$$\omega + \mathrm{dd^c} u_t^\xi = \sigma_\xi \left(\frac{t}{2}\right)^* (\omega + \mathrm{dd^c} u_t).$$

Note that $\{u_t^\xi\}$ is also a subgeodesic segment, and by the cocycle property of E, we have that

$$E(u_t^\xi) - E(u_t) = E(\psi_t^\xi) = \frac{t}{2}\mathrm{Fut}(\mathrm{Re}\, V_\xi), \quad t \in (a, b).$$

Therefore, we have the following.

Lemma 2.6 *Assume that X has vanishing Futaki invariant, then for any subgeodesic segment $\{u_t\} \subset \mathcal{E}^1_\omega$ with $t \in (a, b)$ and $\xi \in N_\mathbb{R}$, one has*

$$E(u_t^\xi) = E(u_t)$$

and

$$L(u_t^\xi) = L(u_t)$$

for any $t \in (a, b)$.

Proof It remains to show the second identity. Observe that

$$\begin{aligned} L(u_t^\xi) &= -\log \int_X e^{h-u_t^\xi} \omega^n = -\log \int_X \sigma_\xi \left(\frac{t}{2}\right)^* (e^{h-u_t}\omega^n) \\ &= -\log \int_X e^{h-u_t}\omega^n = L(u_t), \end{aligned}$$

which completes the proof. □

For later use, we set

$$J_\mathbb{T}\{u_t\} := \inf_{\xi \in N_\mathbb{Q}} J\{u_t^\xi\} \tag{13}$$

for any sublinear subgeodesic ray $\{u_t\} \subset \mathcal{E}^1_\omega$.

Next, we give a useful geometric interpretation of $\{u_t^\xi\}$, in the case when $\xi \in N_\mathbb{Z}$ and $\{u_t\}$ is $\mathbb{T}_\mathbb{R}$-invariant with $u_t \le 0$ for all $t \ge 0$, which will also explain why we added a factor $\frac{1}{2}$ in the above construction.

Note that

$$\Omega := p^*(e^h \omega^n) \wedge \sqrt{-1}\mathrm{d}z \wedge \mathrm{d}\bar{z}$$

is a smooth positive volume form on $X_{\mathbb{C}}$, where $p : X_{\mathbb{C}} \to X$ denotes the projection. Let

$$\Psi^{\xi}(x, z) := \psi^{\xi}_{-\log|z|^2}, \quad U(x, z) := u_{-\log|z|^2} \quad \text{and} \quad U^{\xi}(x, z) := u^{\xi}_{-\log|z|^2},$$

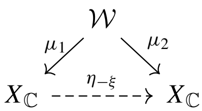
for $(x, z) \in X \times \Delta^* \subset X_{\mathbb{C}}$. Consider the birational map $\eta_{-\xi} : X_{\mathbb{C}} \dashrightarrow X_{\mathbb{C}}$. Under the change of coordinate

$$\frac{t}{2} = -\log|z|,$$

we observe that

$$\Psi^{\xi} = -\log \frac{\eta^*_{-\xi}(\Omega)}{\Omega} \quad \text{and} \quad U^{\xi} = \eta^*_{-\xi} U + \Psi^{\xi} \quad \text{on } X \times \Delta^*. \tag{14}$$

As in [19, (116)], let $\mathcal{W}$ be a resolution of the birational map $\eta_{-\xi}$



Pulling everything back to $\mathcal{W}$, we conclude that

$$\mu_1^* \Psi^{\xi} = -\log \frac{\mu_2^* \Omega}{\mu_1^* \Omega},$$
$$\mu_1^* U^{\xi} = \mu_2^* U + \mu_1^* \Psi^{\xi}.$$

Since we assumed $u_t \le 0$ and $u^{\xi}_t \le 0$, so U and U^{ξ} are well-defined qpsh functions on $X \times \Delta$. However, Ψ^{ξ} might not be so. However, the above identity suggests that, when viewed on $\mathcal{W}$, it is the difference of two qpsh functions. To be more precise, let us further fix a smooth positive volume form $\Omega_{\mathcal{W}}$ on $\mathcal{W}$. Put

$$\Psi^{\xi}_i := \log \frac{\mu_i^* \Omega}{\Omega_{\mathcal{W}}}, \quad i = 1, 2.$$

Then, Ψ^{ξ}_1 and Ψ^{ξ}_2 are globally defined qpsh functions on $\mathcal{W}$ (as μ_i is obtained by a sequence of blow-ups), and we have that

$$\mu_1^* \Psi^{\xi} = \Psi^{\xi}_1 - \Psi^{\xi}_2.$$

Thus, the equality of qpsh functions

$$\mu_1^* U^{\xi} + \Psi^{\xi}_2 = \mu_2^* U + \Psi^{\xi}_1, \tag{15}$$

holds wherever these functions are defined. Let $\mathcal{V}$ be the divisorial valuation on $\mathcal{W}$, such that $(\mu_1)_*\mathcal{V} = G(v)$. By the additivity of Lelong numbers, one has

$$\mathcal{V}(\mu_1^* U^\xi) + \mathcal{V}(\Psi_2^\xi) = \mathcal{V}(\mu_2^* U) + \mathcal{V}(\Psi_1^\xi).$$

On the other hand, since μ_1 is obtained by a sequence of blow-ups, one has

$$K_{\mathcal{W}} - \mu_1^* K_{X_{\mathbb{C}}} = \sum_j a_j E_j,$$

where $a_j \in \mathbb{N}$ and E_j are μ_1-exceptional prime divisors on $\mathcal{W}$. Therefore, we can write

$$\Psi_1^\xi = \log \prod_j |f_j|^{2a_j} + O(1),$$

where $f_j \in \mathcal{O}_{\mathcal{W}}$ are local defining functions for E_j. Thus, the Lelong number of Ψ_1^ξ along any prime divisor F over $\mathcal{W}$ satisfies

$$\nu(\Psi_1^\xi, F) = \mathrm{ord}_F\left(\sum_j a_j E_j\right) = \mathrm{ord}_F\left(K_{\mathcal{W}} - \mu_1^* K_{X_{\mathbb{C}}}\right).$$

The same identity holds for Ψ_2^ξ and μ_2 as well. Therefore, we deduce that

$$\mathcal{V}(\Psi_i^\xi) = \mathcal{V}(K_{\mathcal{W}} - \mu_i^* K_{X_{\mathbb{C}}}), \quad i = 1, 2. \tag{16}$$

Therefore, from (9), we obtain that

$$\begin{aligned} G(v^\xi)(U) &= (\eta_{-\xi})_*(\mu_1)_*\mathcal{V}(U) = (\mu_2)_*\mathcal{V}(U) = \mathcal{V}(\mu_2^* U) \\ &= \mathcal{V}(\mu_1^* U^\xi) + \mathcal{V}(\mu_1^* K_{X_{\mathbb{C}}} - \mu_2^* K_{X_{\mathbb{C}}}), \end{aligned}$$

namely

$$G(v^\xi)(U) = G(v)(U) + \mathcal{V}(\mu_1^* K_{X_{\mathbb{C}}} - \mu_2^* K_{X_{\mathbb{C}}}). \tag{17}$$

Therefore, we obtain the following useful identity. As we will see in (25), it also holds for log Fano pairs, generalizing [19, (111)].

Lemma 2.7 *For any subgeodesic ray* $\{u_t\} \subset \mathcal{E}_\omega^{1,\mathbb{T}_{\mathbb{R}}}$ *with* $u_t \le 0$, $\xi \in N_{\mathbb{Z}}$ *and* $v \in X_{\mathbb{T}}^{\mathrm{div}}$, *one has*

$$A_X(v) - G(v)(U^\xi) = A_X(v^\xi) - G(v^\xi)(U).$$

Proof This follows directly from (10) and (17). □

3 The Proof for the Smooth Case

Let X be a Fano manifold as in the introduction. We prove Theorem 1.1 in this section. We first treat the case when $\mathbb{G} = \{1\}$, since the proof is rather short in view of [12].

Theorem 3.1 *Assume that $\delta(X) > 1$, then X admits a KE metric.*

Proof It suffices to show the properness of D. If D is not proper, then by [12, Theorem 5.3], we have a non-trivial destabilizing geodesic ray $\{u_t\} \subset \mathcal{E}^1_\omega$ satisfying

$$D\{u_t\} \leq 0.$$

By (2), we see that

$$\sup\left\{\tau : \int_X e^{-\hat{u}_\tau}\omega^n < \infty\right\} \leq E\{u_t\}.$$

This implies that (by [12, Lemma 2.4])

$$c[\hat{u}_{E\{u_t\}}] \leq 1,$$

where $c[\cdot]$ denotes the complex singularity exponent (see [12, §2.2] for the precise definition). Therefore, by [12, Theorem 2.3]

$$\inf_F \frac{A_X(F)}{\nu(\hat{u}_{E\{u_t\}}, F)} \leq 1.$$

However, $\nu(\hat{u}_{E\{u_t\}}, F) \leq S_X(F)$ by [12, Proposition 4.5], so that

$$\inf_F \frac{A_X(F)}{\nu(\hat{u}_{E\{u_t\}}, F)} \geq \inf_F \frac{A_X(F)}{S_X(F)} = \delta(X) > 1,$$

which is a contradiction. □

The proof for the case when $\mathbb{G}$ is non-trivial is more involved, which relies on some additional estimates to be shown below. The next result strengthens [12, Proposition 4.5].

Proposition 3.2 *For any sublinear subgeodesic ray $\{u_t\}_t \subset \mathcal{E}^1_\omega$ with $\tau^+_{\hat{u}} > E\{u_t\}$, any prime divisor $F \subset Y \xrightarrow{\pi} X$ over X and any $\lambda \in [0, 1)$, one has*

$$\nu(\hat{u}_{(1-\lambda)E\{u_t\}+\lambda\tau^+_{\hat{u}}}, F) \leq (1-\lambda)S_X(F) + \lambda T_X(F).$$

Here, $T_X(F)$ denotes the pseudoeffective threshold

$$T_X(F) := \sup\{x > 0 \mid -\pi^*K_X - xF \text{ is big}\} = \sup\{\nu(u, F) \mid u \in \mathrm{PSH}_\omega\}. \quad (18)$$

As in [12], the above proposition actually holds for any finite energy sublinear subgeodesic ray in transcendental big classes of compact Kähler manifolds.

Proof When $\lambda = 0$, this is exactly [12, Proposition 4.5]. The general case follows from a similar argument. We give details for the reader's convenience.

Put for simplicity

$$a := (1-\lambda)E\{u_t\} + \lambda\tau_{\hat{u}}^+$$

and

$$f(\tau) := \nu(\hat{u}_\tau, F), \quad \tau \in (-\infty, \tau_{\hat{u}}^+).$$

Then, f is a non-negative, non-decreasing convex function on $(-\infty, \tau_{\hat{u}}^+)$. Moreover, by (18), one has

$$f(\tau) \le T_X(F), \quad \tau \in (-\infty, \tau_{\hat{u}}^+). \tag{19}$$

If $f(-\infty) := \lim_{\tau\to-\infty} f(\tau)$ satisfies $f(-\infty) = f(a)$, then $\nu(\hat{u}_\tau, F) = f(a)$ for all $\tau \in (-\infty, a]$. This implies that $\int_X \omega_{\hat{u}_\tau}^n \le \mathrm{vol}(\{\pi^*\omega\} - f(a)\{F\})$ for $\tau \in (-\infty, a]$, by [12, (37)]. Thus, by (3), we have that

$$\begin{aligned} a &= \frac{1-\lambda}{V}\int_{-\infty}^{\tau_{\hat{u}}^+}\left(\int_X (\omega_{\hat{u}_\tau}^n - \omega^n)\right)\mathrm{d}\tau + \tau_{\hat{u}}^+ \\ &\le \frac{1-\lambda}{V}\int_{-\infty}^{a}\big(\mathrm{vol}(-\pi^*K_X - f(a)F) - \mathrm{vol}(-K_X)\big)\mathrm{d}\tau + \tau_{\hat{u}}^+. \end{aligned}$$

This implies that $\mathrm{vol}(-\pi^*K_X - f(a)F) = \mathrm{vol}(-K_X)$, since $a > -\infty$. Thus, we have that (using (19))

$$\begin{aligned} (1-\lambda)S_X(F) + \lambda T_X(F) &\ge \frac{1-\lambda}{\mathrm{vol}(-K_X)}\int_0^{f(a)} \mathrm{vol}(-K_X)\mathrm{d}x + \lambda f(a) \\ &= (1-\lambda)f(a) + \lambda f(a) = f(a), \end{aligned}$$

what we aimed to prove. So in what follows we assume that $f(-\infty) < f(a)$.

Put

$$b := f'_-(a) = \lim_{h\to 0^+}\frac{f(a) - f(a-h)}{h},$$

which is a positive finite number thanks to the convexity of f. Define

$$g(\tau) := \begin{cases} 0, & \tau \in (-\infty, a - b^{-1}f(a)], \\ b(\tau - a) + f(a), & \tau \in (a - b^{-1}f(a), \tau_{\hat{u}}^+]. \end{cases}$$

Using convexity one more time we have that $g(\tau) \le f(\tau)$ for $\tau \in (-\infty, \tau_{\hat{u}}^+)$. Namely, $v(\hat{u}_\tau, F) \ge g(\tau),\ \tau \in (-\infty, \tau_{\hat{u}}^+)$. This implies that, by [12, (37)] again

$$\int_X \omega_{\hat{u}_\tau}^n \le \mathrm{vol}(-\pi^* K_X - g(\tau)F), \quad \tau \in (-\infty, \tau_{\hat{u}}^+).$$

Thus, using (3), we have that

$$\begin{aligned}
a &= \frac{1-\lambda}{V} \int_{-\infty}^{\tau_{\hat{u}}^+} \left(\int_X (\omega_{\hat{u}_\tau}^n - \omega^n) \right) \mathrm{d}\tau + \tau_{\hat{u}}^+ \\
&\le \frac{1-\lambda}{V} \int_{a-b^{-1}f(a)}^{\tau_{\hat{u}}^+} \big(\mathrm{vol}(-\pi^* K_X - g(\tau)F) - \mathrm{vol}(-K_X)\big) \mathrm{d}\tau + \tau_{\hat{u}}^+ \\
&= \frac{1-\lambda}{\mathrm{vol}(-K_X)b} \int_0^{b(\tau_{\hat{u}}^+ - a)+f(a)} \mathrm{vol}(-\pi^* K_X - xF)\mathrm{d}x - (1-\lambda)(\tau_{\hat{u}}^+ - a + b^{-1} f(a)) + \tau_{\hat{u}}^+ \\
&\le \frac{1-\lambda}{\mathrm{vol}(-K_X)b} \int_0^{T_X(F)} \mathrm{vol}(-\pi^* K_X - xF)\mathrm{d}x + \lambda\tau_{\hat{u}}^+ + (1-\lambda)a - (1-\lambda)b^{-1} f(a).
\end{aligned}$$

This implies that

$$(1-\lambda)b^{-1} f(a) \le (1-\lambda)b^{-1} S_X(F) + \lambda(\tau_{\hat{u}}^+ - a).$$

To estimate $\lambda(\tau_{\hat{u}}^+ - a)$, we use that $g(\tau) \le f(\tau) \le T_X(F)$ (recall (19)), so that

$$g(\tau_{\hat{u}}^+) = b(\tau_{\hat{u}}^+ - a) + f(a) \le T_X(F).$$

This implies that

$$\lambda(\tau_{\hat{u}}^+ - a) \le \lambda b^{-1}(T_X(F) - f(a)).$$

Therefore, we finally arrive at

$$(1-\lambda)b^{-1} f(a) \le (1-\lambda)b^{-1} S_X(F) + \lambda b^{-1}(T_X(F) - f(a)),$$

that is

$$f(a) \le (1-\lambda)S_X(F) + \lambda T_X(F).$$

This completes the proof. □

Another important ingredient is the following result. As we shall see in Proposition 4.2, it also holds for log Fano pairs, generalizing [19, Lemma 3.5].

Proposition 3.3 *For any sublinear subgeodesic ray* $\{u_t\}_t \subset \mathcal{E}^{1,\mathbb{T}_\mathbb{R}}_\omega$, *any* $v \in X^{\mathrm{div}}_{\mathbb{T}}$ *and any* $\xi \in N_\mathbb{Q}$, *one has*

$$A_X(v) + \sup_{\tau\in\mathbb{R}} \left\{\tau - v(\hat{u}^\xi_\tau)\right\} = A_X(v^\xi) + \sup_{\tau\in\mathbb{R}} \left\{\tau - v^\xi(\hat{u}_\tau)\right\}.$$

When $v^\xi = 0$, we put $\sup_{\tau\in\mathbb{R}}\{\tau - v^\xi(\hat{u}_\tau)\} = \tau^+_{\hat{u}}$ (cf. Lemma 2.3).

Proof We first assume that $\xi \in N_\mathbb{Z}$. After subtracting Ct from u_t, we can assume that $u_t \le 0$ and $u^\xi_t \le 0$ for all $t \ge 0$, so that Lemma 2.3 is applicable. Let U and U^ξ denote the qpsh functions on $X \times \Delta$ corresponding to $\{u_t\}$ and $\{u^\xi_t\}$, respectively. In view of Lemma 2.3, it amounts to showing that

$$A_X(v) - G(v)(U^\xi) = A_X(v^\xi) - G(v^\xi)(U),$$

which is exactly Lemma 2.7. Therefore, we conclude when $\xi \in N_\mathbb{Z}$.

The case for $\xi \in N_\mathbb{Q}$ follows from a scaling argument. Indeed, let $k \in \mathbb{N}$ be such that $k\xi \in N_\mathbb{Z}$. For any $v = v_{\mu,\zeta} \in X^{\mathrm{div}}_{\mathbb{T}}$, one has [recall (8)]

$$(kv)^{k\xi}\left(\sum_\alpha f_\alpha 1^\alpha\right) = \min_\alpha\{k\mu(f_\alpha) + \langle\alpha, k\zeta + k\xi\rangle\} = kv^\xi\left(\sum_\alpha f_\alpha 1^\alpha\right).$$

This implies that

$$v^\xi = k^{-1}(kv)^{k\xi}.$$

Consider the rescaled ray

$$\phi_t := u_{kt}.$$

Then

$$\hat{\phi}_{k\tau} = \hat{u}_\tau, \quad \phi^{k\xi}_t = u^\xi_{kt} \quad \text{and} \quad \hat{\phi}^{k\xi}_{k\tau} = \hat{u}^\xi_\tau.$$

Therefore, we derive that

$$\begin{aligned} A_X(v^\xi) + \sup_{\tau\in\mathbb{R}} \left\{\tau - v^\xi(\hat{u}_\tau)\right\} &= k^{-1}\left(A_X((kv)^{k\xi}) + \sup_{\tau\in\mathbb{R}}\left\{k\tau - (kv)^{k\xi}(\hat{\phi}_{k\tau})\right\}\right) \\ &= k^{-1}\left(A_X(kv) + \sup_{\tau\in\mathbb{R}}\left\{k\tau - (kv)(\hat{\phi}^{k\xi}_{k\tau})\right\}\right) \\ &= A_X(v) + \sup_{\tau\in\mathbb{R}}\left\{\tau - v(\hat{u}^\xi_\tau)\right\}. \end{aligned}$$

This completes the proof. □

We also need the following key result, showing that D is geodesic stable if $\delta^r(X) > 1$. We will see in Theorem 4.3 that the same holds for log Fano pairs, which strengthens [19, Theorem 1.3].

Theorem 3.4 *Assume that X has vanishing Futaki invariant and $\delta^r(X) > 1$, then there exists $\lambda > 0$, such that for any sublinear subgeodesic ray $\{u_t\}_t \subset \mathcal{E}_\omega^{1,\mathbb{T}_\mathbb{R}}$*

$$D\{u_t\} \geq \lambda J_\mathbb{T}\{u_t\}.$$

Proof If for some $\xi \in N_\mathbb{Q}$, one has $J\{u_t^\xi\} = 0$, i.e., $\{u_t^\xi\}$ is trivial, then what we aimed to prove holds trivially, as $D\{u_t\} = D\{u_t^\xi\} = J\{u_t^\xi\} = 0$ [by Lemma 2.6 and (2)]. Therefore, we assume that $J\{u_t^\xi\} = \tau^+_{\hat{u}^\xi} - E\{u_t^\xi\} > 0$ for any $\xi \in N_\mathbb{Q}$.

Choose $\delta \in (1, \delta^r(X))$. By (7), we can find a sequence $v_k \in X_\mathbb{T}^{\mathrm{div}}$, such that

$$L\{u_t\} \geq A_X(v_k) + \sup_{\tau \in \mathbb{R}} \left\{\tau - v_k(\hat{u}_\tau)\right\} - \frac{1}{k}.$$

If it happens that $v_k^{-\xi_k} = 0$ for some $\xi_k \in N_\mathbb{Q}$, then by Proposition 3.3 and Lemma 2.6

$$\begin{aligned} D\{u_t\} &\geq A_X(v_k^{-\xi_k}) + \sup_{\tau \in \mathbb{R}} \left\{\tau - v_k^{-\xi_k}(\hat{u}_\tau^{\xi_k})\right\} - E\{u_t^{\xi_k}\} - \frac{1}{k} \\ &= \tau^+_{\hat{u}^{\xi_k}} - E\{u_t^{\xi_k}\} - \frac{1}{k} = J\{u_t^{\xi_k}\} - \frac{1}{k}. \end{aligned}$$

Otherwise there exists $\xi_k \in N_\mathbb{Q}$, such that

$$A_X(v_k^{-\xi_k}) \geq \delta S_X(v_k^{-\xi_k}).$$

Hence

$$\begin{aligned} D\{u_t\} = L\{u_t\} - E\{u_t\} &\geq A_X(v_k) + \sup_{\tau \in \mathbb{R}} \left\{\tau - v_k(\hat{u}_\tau)\right\} - E\{u_t\} - \frac{1}{k} \\ &= A_X(v_k^{-\xi_k}) + \sup_{\tau \in \mathbb{R}} \left\{\tau - v_k^{-\xi_k}(\hat{u}_\tau^{\xi_k})\right\} - E\{u_t^{\xi_k}\} - \frac{1}{k} \\ &\geq \delta S_X(v_k^{-\xi_k}) + \sup_{\tau \in \mathbb{R}} \left\{\tau - v_k^{-\xi_k}(\hat{u}_\tau^{\xi_k})\right\} - E\{u_t^{\xi_k}\} - \frac{1}{k}. \end{aligned}$$

In the second equality, we used Proposition 3.3 and Lemma 2.6.

Set

$$\lambda := \frac{\delta - 1}{n} \quad \text{and} \quad \tau_0 := (1 - \lambda) E\{u_t^{\xi_k}\} + \lambda \tau^+_{\hat{u}^{\xi_k}}.$$

We can assume $\lambda \in (0, 1)$. Then, we deduce that

$$\begin{aligned} D\{u_t\} &\geq \delta S_X(v_k^{-\xi_k}) + \tau_0 - v_k^{-\xi_k}(\hat{u}_{\tau_0}^{\xi_k}) - E\{u_t^{\xi_k}\} - \frac{1}{k} \\ &= \delta S_X(v_k^{-\xi_k}) - v_k^{-\xi_k}(\hat{u}_{\tau_0}^{\xi_k}) + \lambda\big(\tau_{\hat{u}^{\xi_k}}^{+} - E\{u_t^{\xi_k}\}\big) - \frac{1}{k} \\ &\geq \delta S_X(v_k^{-\xi_k}) - (1-\lambda) S_X(v_k^{-\xi_k}) - \lambda T_X(v_k^{-\xi_k}) + \lambda J\{u_t^{\xi_k}\} - \frac{1}{k} \\ &= \frac{(\delta-1)(n+1)}{n} S_X(v_k^{-\xi_k}) - \frac{\delta-1}{n} T_X(v_k^{-\xi_k}) + \lambda J\{u_t^{\xi_k}\} - \frac{1}{k} \\ &\geq \lambda J\{u_t^{\xi_k}\} - \frac{1}{k} \geq \lambda J_{\mathbb{T}}\{u_t\} - \frac{1}{k}. \end{aligned}$$

In the second inequality, we used Proposition 3.2 and (4). To get the last line, we used the fact that $S_X(v) \geq \frac{T_X(v)}{n+1}$ for any $v \in X^{\mathrm{div}}$ (see, e.g., [6, (3.1)]) and (13). This completes the proof. □

Finally, we are in the position to prove our main theorem.

Theorem 3.5 (=Theorem 1.1) *Let X be a Fano manifold with vanishing Futaki invariant. If $\delta^r(X) > 1$, then X admits a KE metric.*

Proof We only need to show that D is proper modulo $\mathbb{T}$.

Assume that D is not proper modulo $\mathbb{T}$, then there exists a sequence $\phi_j \in \mathcal{E}_\omega^{1,\mathbb{T}_\mathbb{R}}$, such that

$$D(\phi_j) \leq \frac{1}{j} J_{\mathbb{T}}(\phi_j) - j, \quad j \in \mathbb{N}.$$

By [19, Lemma 2.15], we can assume that $\sup \phi_j = 0$ and

$$J(\phi_j) = J_{\mathbb{T}}(\phi_j). \tag{20}$$

Using $J(\phi_j) \leq -E(\phi_j)$ and $D(\phi_j) = L(\phi_j) - E(\phi_j)$, we obtain that

$$L(\phi_j) \leq \left(1 - \frac{1}{j}\right) E(\phi_j) - j.$$

Let

$$T_j := d_1(0, \phi_j)$$

denote the d_1-distance from 0 to ϕ_i. We claim that

$$T_j \to \infty.$$

If not, there would exist $A > 0$ and a subsequence of $\{\phi_j\}$, such that $D(\phi_j) \leq \frac{1}{j} d_1(0, \phi_j) - j$ and $d_1(0, \phi_j) \leq A$. Up to further passing to a subsequence, we get an

L^1-limit $\phi_\infty \in \mathcal{E}^1_\omega$ of ϕ_j with $-\infty < D(\phi_\infty) \leq \liminf_j D(\phi_j) = -\infty$, which is a contradiction.

Next, let $[0, T_j] \ni t \mapsto u^j_t \in \mathcal{E}^{1,\mathbb{T}_\mathbb{R}}_\omega$ be the unit speed finite energy $\mathbb{T}_\mathbb{R}$-invariant geodesic segment joining 0 and ϕ_j, so we have that

$$\sup u^j_t = 0, \quad u^j_{T_j} = \phi_j, \quad E(u^j_t) = -t, \quad t \in [0, T_j].$$

By the convexity of L [5], we also have that

$$\frac{L(u^j_t)}{t} \leq -1 + \frac{1}{j}, \quad t \in [0, T_j].$$

Then, arguing as in the proof of [12, Theorem 5.3], we obtain a subgeodeisc ray $\{u_t\} \subset \mathcal{E}^{1,\mathbb{T}_\mathbb{R}}_\omega$ from the L^1-compactness of $\{u^j_t\}$, which satisfies

$$\sup u_t = 0, \quad L\{u_t\} \leq -1, \quad E\{u_t\} \geq -1, \quad D\{u_t\} \leq 0.$$

Moreover, the ray $\{u_t\}$ is non-trivial, in the sense that

$$a := J\{u_t\} = \tau^+_{\hat{u}} - E\{u_t\} = -E\{u_t\} > 0.$$

Indeed, consider the maximization $\{v_t\}$ of $\{u_t\}$ defined as in [12, (30)]. As shown in the last paragraph of the proof of [12, Theorem 5.3], the ray $\{v_t\}$ is non-trivial. Thus, $J\{v_t\} = \tau^+_v - E\{v_t\} > 0$. By [12, (30)], one has $\tau^+_u = \tau^+_v$. While by [12, Proposition 3.13], one has $E\{u_t\} = E\{v_t\}$. So $J\{u_t\} = J\{v_t\} > 0$, as claimed.

Note that $(0, \infty) \ni t \mapsto E(u_t)$ is convex, so we derive that

$$E(u_t) \leq -at, \quad t \geq 0.$$

Next, for any $\xi \in N_\mathbb{Q}$, we claim that

$$J\{u^\xi_t\} \geq a. \tag{21}$$

This will imply that $J_\mathbb{T}\{u_t\} \geq a > 0$ with $D\{u_t\} \leq 0$, contradicting Theorem 3.4. Thus, we complete the proof.

Therefore, it remains to show (21). Recall that from the proof of [12, Theorem 5.3], for any $t \in (0, \infty) \backslash F$, one has

$$u^j_t \xrightarrow{L^1(\omega^n)} u_t,$$

where $F \subset (0, \infty)$ is a set of Lebesgue measure zero. This also implies that

$$u^{j,\xi}_t \xrightarrow{L^1(\omega^n)} u^\xi_t, \quad t \in (0, \infty) \backslash F,$$

thanks to the definition (12). Also note that, by Lemma 2.6

$$E(u_t^{j,\xi}) = E(u_t^j) = -t, \quad E(u_t^\xi) = E(u_t).$$

Therefore, we deduce that

$$\begin{aligned} J(u_t^\xi) &= \frac{1}{V}\int_X u_t^\xi \omega^n - E(u_t^\xi) = \lim_j \frac{1}{V}\int_X u_t^{j,\xi}\omega^n - E(u_t) \\ &= \lim_j \left(\frac{1}{V}\int_X u_t^{j,\xi}\omega^n - E(u_t^{j,\xi})\right) - t - E(u_t) \\ &= \lim_j J(u_t^{j,\xi}) - t - E(u_t) \geq \lim_j J(u_t^{j,\xi}) - t + at. \end{aligned}$$

This holds for any $t \in (0,\infty)\backslash F$. Now, applying Lemma 2.2, we further deduce that

$$\begin{aligned} J(u_t^\xi) &\geq \liminf_j \frac{t}{T_j} J(u_{T_j}^{j,\xi}) - t + at - C_0 \\ &\geq \liminf_j \frac{t}{T_j} J(\phi_j) - t + at - C_0 \\ &\geq \liminf_j \frac{t}{T_j}(-E(\phi_j) - C_0) - t + at - C_0. \end{aligned}$$

In the second inequality, we used (20) and in the last inequality we used (1). Therefore, we obtain that

$$J(u_t^\xi) \geq \liminf_j \frac{t}{T_j}(T_j - C_0) - t + at - C_0 = at - C_0, \quad t \in (0,\infty)\backslash F.$$

This implies that $J\{u_t^\xi\} \geq a$, as claimed. Therefore, we complete the proof. □

4 The Singular Case

In this part, we show that our proof naturally extends to the setting of log Fano varieties. In [12, §6.1], we have already shown how to treat the case of discrete automorphism groups. To deal with continuous automorphisms, we need to extend the arguments in the previous section to the singular setting.

To be more precise, let (Z, Δ) be a log Fano variety. Namely, Z is a normal projective variety and Δ an effective Weil $\mathbb{Q}$-divisor on Z, such that

$$L := -K_Z - \Delta$$

is an ample $\mathbb{Q}$-Cartier divisor and that (Z, Δ) has klt singularities.

Let $\mathbb{G} := \mathrm{Aut}(Z, \Delta)$ be the group of automorphisms of Z that preserve Δ. We assume in what follows that $\mathbb{G}$ is reductive. Let $\mathbb{T}$ be a maximal torus of $\mathbb{G}$. As in the smooth case, assume that $\mathbb{T} = (\mathbb{C}^*)^r$, $\mathbb{T}_{\mathbb{R}} := (S^1)^r$, $N_{\mathbb{Z}} := \mathrm{Hom}(\mathbb{C}^*, \mathbb{T})$, $N_{\mathbb{Q}} := N_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$, and $N_{\mathbb{R}} := N_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$.

The reduced delta invariant of (Z, Δ) is defined by (following [19, Theorem 1.3] and [31, Appendix A])

$$\delta^r(Z, \Delta) := \inf_{v \in Z^{\mathrm{div}}_{\mathbb{T}}} \sup_{\xi \in N_{\mathbb{Q}}} \frac{A_{Z,\Delta}(v^{\xi})}{S_L(v^{\xi})},$$

where the inf is over all $\mathbb{T}$-invariant divisorial valuations v on Z, and v^{ξ} is the twist of v [defined as in (8)]. Here, $A_{Z,\Delta}(\cdot)$ denotes the log discrepancy with respect to the pair (Z, Δ) and $S_L(\cdot)$ denotes the expected vanishing order (see, e.g., [24, §2.1] for the precise definition). Again, if it happens that $v^{\xi} = 0$, then we put $\frac{A_{Z,\Delta}(v^{\xi})}{S_L(v^{\xi})} = +\infty$.

For the log Fano pair (Z, Δ), one can also define Futaki invariant as in [15]; see, e.g., [16, §4.1] for a general definition.

We will give a simplified proof of the following result of Li [19, Theorem 1.3].

Theorem 4.1 *Let (Z, Δ) be a log Fano variety with vanishing Futaki invariant and $\delta^r(Z, \Delta) > 1$, then (Z, Δ) admits a KE metric.*

The proof follows the exactly same strategy as we did for Theorem 3.5. To give more details, first fix a smooth $\mathbb{T}_{\mathbb{R}}$-invariant Hermitian metric H on L whose Chern curvature form is a smooth Kähler form ω on Z.

Such a Hermitian metric can be explicitly constructed as follows. Choose $l_0 > 0$ sufficiently divisible, such that $l_0 L$ is a very ample line bundle on Z. Note that the $\mathbb{T}$-action on Z lifts to L, and hence, $\mathbb{T}$ acts on $H^0(X, l_0 L)$, as well. Then, we can pick a basis $\{s_i\}$ of $H^0(X, l_0 L)$, such that each s_i spans a one-dimensional $\mathbb{T}$-invariant subspace. Let

$$H := \left(\sum_i |s_i|^2 \right)^{-\frac{1}{l_0}}, \tag{22}$$

which is clearly a $\mathbb{T}_{\mathbb{R}}$-invariant Hermitian metric on L, whose Chern curvature form is $\frac{1}{l_0}$ times the pull back of the Fubini–Study metric induced by the embedding of the basis $\{s_i\}$.

Following [3, Definition 3.1], H induces an *adapted* measure μ_H on Z as follows. Choose $r > 0$ such that $r(K_X + \Delta)$ is Cartier. Let σ be a local non-vanishing section of $r(K + D)$. Then

$$\mu_H := \frac{(\sqrt{-1}^{rn^2} \sigma \wedge \bar{\sigma})^{1/r}}{(H^{-r}(\sigma, \sigma))^{1/r}}.$$

Note that μ_H does not depend on the choice of σ and hence is a globally defined measure on Z. And (Z, Δ) being klt means exactly that μ_H has finite total measure on Z. If Z is a smooth Fano manifold and $\Delta = 0$, then μ_H is (up to a constant multiple) simply equal to the volume form $e^h \omega^n$ in (11).

Next, choose a $\mathbb{T}$-equivariant log resolution of (Z, Δ), $\pi : X \to Z$, such that

$$K_X + \sum_i a_i E_i = \pi^*(K_Z + \Delta).$$

Here, $a_i < 1$ as (Z, Δ) is klt and each E_i is $\mathbb{T}$-invariant. We put

$$Q := \sum_i a_i E_i.$$

The measure μ_H lifts to a measure on X via the birational map π, which we denote by μ for short, and we can write (see [3, Lemma 3.2])

$$\mu = e^{\chi - \psi} dV,$$

where χ, ψ are qpsh functions on X with analytic singularities that are determined by the divisor Q and dV is a smooth positive volume form on X. Let

$$\theta := \pi^*\omega,$$

which is a smooth non-negative closed $(1, 1)$-form on X, representing the big class $c_1(\pi^*L)$. Therefore, we are now in the situation investigated in [12].

Note that the reduced delta invariant $\delta^r(Z, \Delta)$ satisfies

$$\delta^r(Z, \Delta) = \delta^r(X, Q) := \inf_{v \in X_{\mathbb{T}}^{\mathrm{div}}} \sup_{\xi \in N_{\mathbb{Q}}} \frac{A_{\chi,\psi}(v^\xi)}{S_{\pi^*L}(v^\xi)},$$

where $A_{\chi,\psi}(\cdot) = A_{X,Q}(\cdot)$ is the log discrepancy defined with respect to the pair (X, Q). More precisely, for any prime divisor F over X, we let

$$A_{\chi,\psi}(F) := A_X(F) + \nu(\chi, F) - \nu(\psi, F) = A_X(F) - \mathrm{ord}_F(Q).$$

For a general divisorial valuation $v = \lambda\, \mathrm{ord}_F$, let

$$A_{\chi,\psi}(v) := \lambda A_{\chi,\psi}(F).$$

As in [12], consider the Ding functional

$$D_\mu(u) := L_\mu(u) - E_\theta(u), \quad u \in \mathcal{E}^1(X, \theta),$$

where

$$L_\mu(u) := -\log \int_X e^{-u} \mathrm{d}\mu,$$
$$E_\theta(u) := \frac{1}{(n+1)V} \int_X u \sum_{i=0}^n \theta^i \wedge \theta_u^{n-i}.$$

Here, $\mathcal{E}^1(X,\theta)$ denotes the set of finite energy θ-psh functions and $V := \int_X \theta^n = \mathrm{vol}(-K_X - \Delta)$. Also, put

$$J(u) := J(\theta, \theta_u) := \frac{1}{V}\int_X u\theta^n - E_\theta(u),$$

and

$$J_{\mathbb{T}}(u) := \inf_{\sigma\in\mathbb{T}} J(\theta, \sigma^*\theta_u).$$

By the variational principle [3, 9, 17] (see [19, Theorem 2.18] for the precise statement that we need here), our goal is to show that D_μ is proper modulo $\mathbb{T}$, that is

$$D_\mu(u) \geq \varepsilon J_{\mathbb{T}}(u) - C, \quad u \in \mathcal{E}^1(X,\theta)^{\mathbb{T}_{\mathbb{R}}} \tag{23}$$

for some constants $\varepsilon > 0, C > 0$, under the assumption that (Z, Δ) has vanishing Futaki invariant and $\delta^r(Z,\Delta) > 1$. Here, $\mathcal{E}^1(X,\theta)^{\mathbb{T}_{\mathbb{R}}}$ denotes the set of $\mathbb{T}_{\mathbb{R}}$-invariant elements in $\mathcal{E}^1(X,\theta)$.

To achieve (23), we will use subgeodesic rays and their twists, as in the smooth case. More precisely, for any $\xi \in N_{\mathbb{R}}$, it determines a holomorphic vector field V_ξ on X, let $\{\sigma_\xi(t)\}_{t\in\mathbb{R}}$ be the one-parameter subgroup of $\mathbb{T}$ generated by $\mathrm{Re}\, V_\xi$. For any subgeodesic segment $\{u_t\} \subset \mathcal{E}^1(X,\theta)$ with $t \in (a,b), 0 \leq a < b \leq \infty$, one can twist the segment using ξ by putting

$$u_t^\xi := -\log\frac{\sigma_\xi(t/2)^*(e^{-u_t}\mu)}{\mu} = \sigma_\xi\left(\frac{t}{2}\right)^* u_t + \psi_t^\xi, \quad t \in (a,b),$$

where

$$\psi_t^\xi := -\log\frac{\sigma_\xi(t/2)^*\mu}{\mu}.$$

Observe that ψ_ξ^t is a smooth function globally defined on X. Indeed, letting $\hat{s}_i := \pi^* s_i \in H^0(X, \pi^*(l_0 L))$, where $\{s_i\}$ is the basis chosen in (22), then from the definition of μ (and μ_H), it is clear that

$$\psi_t^\xi = \frac{1}{l_0}\log\frac{\sum_i |\sigma_\xi(t/2)^*\hat{s}_i|^2}{\sum_i |\hat{s}_i|^2}. \tag{24}$$

From here, it also follows that:

$$\theta + \mathrm{dd^c}\psi_t^\xi = \sigma_\xi\left(\frac{t}{2}\right)^*\theta, \quad \theta + \mathrm{dd^c}u_t^\xi = \sigma_\xi\left(\frac{t}{2}\right)^*\theta_{u_t}.$$

Therefore, $\{u_t^\xi\}$ is also a subgeodesic segment in $\mathcal{E}^1(X,\theta)$. Since we assumed that (Z,Δ) has vanishing Futaki invariant, then as in Lemma 2.6, one has

$$E_\theta(u_t^\xi) - E_\theta(u_t) = E_\theta(\psi_t^\xi) = \frac{1}{2}\mathrm{Fut}(\mathrm{Re}\, V_\xi) = 0 \quad \text{and} \quad L_\mu(u_t^\xi) = L_\mu(u_t).$$

Moreover, one has the following generalization of Proposition 3.3.

Proposition 4.2 *For any sublinear subgeodesic ray $\{u_t\}_t \subset \mathcal{E}_\theta^{1,\mathbb{T}_\mathbb{R}}$, any $v \in X_\mathbb{T}^{\mathrm{div}}$ and any $\xi \in N_\mathbb{Q}$, one has*

$$A_{\chi,\psi}(v) + \sup_{\tau\in\mathbb{R}} \left\{\tau - v(\hat{u}_\tau^\xi)\right\} = A_{\chi,\psi}(v^\xi) + \sup_{\tau\in\mathbb{R}} \left\{\tau - v^\xi(\hat{u}_\tau)\right\}.$$

Proof As in the proof of Proposition 3.3, it suffices to treat the case when $\xi \in N_\mathbb{Z}$, $u_t \le 0$, $u_t^\xi \le 0$, and argue that

$$A_{\chi,\psi}(v) - G(v)(U^\xi) = A_{\chi,\psi}(v^\xi) - G(v^\xi)(U), \tag{25}$$

where U and U^ξ are qpsh functions on $X \times \{|z| < 1\}$ that are associated with $\{u_t\}$ and $\{u_t^\xi\}$, respectively.

Similarly as in Sect. 2.4, let $\mathcal{W}$ be a resolution of the birational map $\eta_{-\xi}$

$$\begin{array}{ccccc} & & \mathcal{W} & & \\ & \swarrow^{\mu_1} & & \searrow^{\mu_2} & \\ X_\mathbb{C} & & \overset{\eta_{-\xi}}{\dashrightarrow} & & X_\mathbb{C} \\ \pi_\mathbb{C}\downarrow & & & & \pi_\mathbb{C}\downarrow \\ Z_\mathbb{C} & & \overset{\eta_{-\xi}}{\dashrightarrow} & & Z_\mathbb{C} \end{array}$$

where $X_\mathbb{C} := X \times \mathbb{C}$, $Z_\mathbb{C} := Z \times \mathbb{C}$ and $\pi_\mathbb{C} := \pi \times \mathrm{id}$. Let $\mathcal{V}$ be the divisorial valuation on $\mathcal{W}$, such that

$$(\mu_1)_*\mathcal{V} = G(v).$$

To prove (25), it suffices to show the following two identities:

$$A_{\chi,\psi}(v^\xi) - A_{\chi,\psi}(v) = \mathcal{V}(\mu_1^*(K_{X_\mathbb{C}} + Q_\mathbb{C}) - \mu_2^*(K_{X_\mathbb{C}} + Q_\mathbb{C})), \tag{26}$$

and

$$G(v^\xi)(U) - G(v)(U^\xi) = \mathcal{V}(\mu_1^*(K_{X_\mathbb{C}} + Q_\mathbb{C}) - \mu_2^*(K_{X_\mathbb{C}} + Q_\mathbb{C})), \tag{27}$$

which generalize (10) and (17) respectively. Here, $Q_\mathbb{C} := Q \times \mathbb{C}$.

Note that

$$A_{\chi,\psi}(v^\xi) - A_{\chi,\psi}(v) = A_{Z,\Delta}(\underline{v}^\xi) - A_{Z,\Delta}(\underline{v}),$$

where $\underline{v}^{\xi} := \pi_*(v^{\xi}) = (\pi_* v)^{\xi}$ and $\underline{v} := \pi_* v$, so (26) follows from the following identity (letting $\Delta_{\mathbb{C}} := \Delta \times \mathbb{C}$):

$$A_{Z,\Delta}(\underline{v}^{\xi}) - A_{Z,\Delta}(\underline{v}) = \mathcal{V}((\pi_{\mathbb{C}} \circ \mu_1)^*(K_{Z_{\mathbb{C}}} + \Delta_{\mathbb{C}}) - (\pi_{\mathbb{C}} \circ \mu_2)^*(K_{Z_{\mathbb{C}}} + \Delta_{\mathbb{C}})),$$

whose proof is exactly the same as the one for (10); see also the proof of [19, Proposition 3.3]. Therefore, we omit the details.

The identity (27) can be argued in a similar way as we did for (17). Indeed, let

$$\Omega := p^*\mu \wedge \sqrt{-1}\mathrm{d}z \wedge \mathrm{d}\bar{z} = e^{\chi\circ p - \psi\circ p} \cdot p^*(\mathrm{d}V) \wedge \sqrt{-1}\mathrm{d}z \wedge \mathrm{d}\bar{z},$$

where $p : X_{\mathbb{C}} \to X$ denotes the projection. Note that Ω is a singular volume form on $X_{\mathbb{C}}$. Away from the divisor $Q_{\mathbb{C}}$, Ω is a smooth positive volume form, but it could have both zeros and poles along $Q_{\mathbb{C}}$ with orders given by the coefficients of $Q_{\mathbb{C}}$.

We also have that (as in (14))

$$\Psi^{\xi} = -\log \frac{\eta^*_{-\xi}\Omega}{\Omega} \quad \text{and} \quad U^{\xi} = \eta^*_{-\xi}U + \Psi^{\xi} \quad \text{on } X \times \{0 < |z| < 1\}.$$

After pulling these functions to $\mathcal{W}$, we obtain that

$$\mu_1^*\Psi^{\xi} = -\log \frac{\mu_2^*\Omega}{\mu_1^*\Omega} \quad \text{and} \quad \mu_1^*U^{\xi} = \mu_2^*U + \mu_1^*\Psi^{\xi}.$$

Here, U and U^{ξ} are qpsh functions defined on $X \times \{|z| < 1\}$, while Ψ^{ξ} is a priori a smooth function (recall (24)) defined on $X \times \mathbb{C}^*$, which could be singular at $X \times \{0\}$. We now show that $\mu_1^*\Psi^{\xi}$ is actually the difference of two qpsh functions defined on $\mathcal{W}$. Indeed, fix a smooth positive volume form $\Omega_{\mathcal{W}}$ on $\mathcal{W}$. Put

$$F_i := \log \frac{\mu_i^*\big(p^*(\mathrm{d}V) \wedge \sqrt{-1}\mathrm{d}z \wedge \mathrm{d}\bar{z}\big)}{\Omega_{\mathcal{W}}}, \quad i = 1, 2.$$

Then, F_1 and F_2 are two globally defined qpsh functions on $\mathcal{W}$. As in (16)

$$\mathcal{V}(F_i) = \mathcal{V}(K_{\mathcal{W}} - \mu_i^* K_{X_{\mathbb{C}}}), \quad i = 1, 2.$$

Moreover, we can write

$$\mu_1^*\Psi^{\xi} = (F_1 + \psi \circ p \circ \mu_2 + \chi \circ p \circ \mu_1) - (F_2 + \psi \circ p \circ \mu_1 + \chi \circ p \circ \mu_2),$$

which is the difference of two qpsh functions on $\mathcal{W}$ as claimed. Therefore, the identity of qpsh functions

$$\mu_1^*U^{\xi} + (F_2 + \psi \circ p \circ \mu_1 + \chi \circ p \circ \mu_2) = \mu_2^*U + (F_1 + \psi \circ p \circ \mu_2 + \chi \circ p \circ \mu_1)$$

holds wherever these functions are defined.

By the additivity of Lelong numbers, we obtain that

$$\mathcal{V}(\mu_2^* U) - \mathcal{V}(\mu_1^* U^\xi) = \mathcal{V}(F_2 + \psi \circ p \circ \mu_1 + \chi \circ p \circ \mu_2) \\ - \mathcal{V}(F_1 + \psi \circ p \circ \mu_2 + \chi \circ p \circ \mu_1).$$

And also, by our choice of χ and ψ

$$\mathcal{V}(\psi \circ p \circ \mu_i) - \mathcal{V}(\chi \circ p \circ \mu_i) = \mathcal{V}(\mu_i^* Q_{\mathbb{C}}), \quad i = 1, 2.$$

Therefore, by (9), we finally arrive at

$$G(v^\xi)(U) - G(v)(U^\xi) = \mathcal{V}(\mu_1^*(K_{X_{\mathbb{C}}} + Q_{\mathbb{C}}) - \mu_2^*(K_{X_{\mathbb{C}}} + Q_{\mathbb{C}})),$$

as claimed. Thus, we complete the proof. □

With these preparations in hand, now we can proceed verbatim, following the lines for Theorems 3.4 and 3.5. We record the following generalization of Theorem 3.4, without giving the proof. In each step of the argument, we just need to replace A_X with $A_{\chi,\psi}$, S_X with $S_{\pi^* L}$, D with D_μ, L with L_μ and E with E_θ.

Theorem 4.3 *Assume that* (Z, Δ) *has vanishing Futaki invariant and* $\delta^r(Z, \Delta) > 1$, *then there exists* $\lambda > 0$, *such that for any sublinear subgeodesic ray* $\{u_t\}_t \subset \mathcal{E}_\theta^{1,\mathbb{T}_\mathbb{R}}$

$$D_\mu\{u_t\} \geq \lambda J_{\mathbb{T}}\{u_t\}.$$

Then, we can prove Theorem 4.1 by arguing in the same way as we did for Theorem 3.5. In each step of the argument, we just need to replace ω with θ, D with D_μ, L with L_μ and E with E_θ. We need to remark that the preliminary results needed in the above proof, such as Lemma 2.2, Proposition 2.4, and Proposition 3.2, hold in the log Fano setting as well (thanks to [19, Lemma 2.4], [11, Theorem 5.7] and [12, §3]). This completes the proof of Theorem 4.1.

Acknowledgements The author is grateful to G. Tian for helpful suggestions in this work. He is also grateful to both G. Tian and X. Zhu for their patient guidance over the years. The author is supported by NSFC Grants 12101052, 12271040, and 12271038.

Data availability The author confirms that the data supporting the findings of this study are available within the article and its references.

Declarations

Conflict of interest The author declares that he has no conflict of interest.

References

1. Altmann, K., Hausen, J., Süss, H.: Gluing affine torus actions via divisorial fans. Transform. Groups **13**(2), 215–242 (2008)

2. Berman, R.J.: K-polystability of $\mathbb{Q}$-Fano varieties admitting Kähler–Einstein metrics. Invent. Math. **203**(3), 973–1025 (2016)
3. Berman, R.J., Boucksom, S., Eyssidieux, P., Guedj, V., Zeriahi, A.: Kähler–Einstein metrics and the Kähler–Ricci flow on log Fano varieties. J. Reine Angew. Math. **751**, 27–89 (2019)
4. Berman, R.J., Boucksom, S., Jonsson, M.: A variational approach to the Yau–Tian–Donaldson conjecture. J. Am. Math. Soc. **34**(3), 605–652 (2021)
5. Berndtsson, B.: A Brunn–Minkowski type inequality for Fano manifolds and some uniqueness theorems in Kähler geometry. Invent. Math. **200**(1), 149–200 (2015)
6. Blum, H., Jonsson, M.: Thresholds, valuations, and K-stability. Adv. Math. **365**, 107062, 57 pp. (2020)
7. Boucksom, S., Hisamoto, T., Jonsson, M.: Uniform K-stability, Duistermaat–Heckman measures and singularities of pairs. Ann. Inst. Fourier (Grenoble) **67**(2), 743–841 (2017)
8. Darvas, T.: Geometric pluripotential theory on Kähler manifolds. In: Advances in Complex Geometry, pp. 1–104. Contemp. Math., 735, Amer. Math. Soc., Providence, RI (2019)
9. Darvas, T., Rubinstein, Y.A.: Tian's properness conjectures and Finsler geometry of the space of Kähler metrics. J. Am. Math. Soc. **30**(2), 347–387 (2017)
10. Darvas, T., Xia, M.: The closures of test configurations and algebraic singularity types. Adv. Math. **397**, Paper No. 108198, 56 pp. (2022)
11. Darvas, T., Xia, M., Zhang, K.: A transcendental approach to non-Archimedean metrics of pseudoeffective classes. arXiv:2302.02541v3 (2024) (To appear in Comment. Math. Helv.)
12. Darvas, T., Zhang, K.: Twisted Kähler–Einstein metrics in big classes. Commun. Pure Appl. Math. **77**(12), 4289–4327 (2024)
13. Fujita, K.: A valuative criterion for uniform K-stability of $\mathbb{Q}$-Fano varieties. J. Reine Angew. Math. **751**, 309–338 (2019)
14. Fujita, K., Odaka, Y.: On the K-stability of Fano varieties and anticanonical divisors. Tohoku Math. J. (2) **70**(4), 511–521 (2018)
15. Futaki, A.: An obstruction to the existence of Einstein Kähler metrics. Invent. Math. **73**(3), 437–443 (1983)
16. Han, J., Li, C.: On the Yau–Tian–Donaldson conjecture for generalized Kähler–Ricci soliton equations. Commun. Pure Appl. Math. **76**(9), 1793–1867 (2023)
17. Hisamoto, T.: Stability and coercivity for toric polarizations. arXiv:1610.07998v3 (2020)
18. Li, C.: K-semistability is equivariant volume minimization. Duke Math. J. **166**(16), 3147–3218 (2017)
19. Li, C.: G-uniform stability and Kähler–Einstein metrics on Fano varieties. Invent. Math. **227**(2), 661–744 (2022)
20. Li, C.: Geodesic rays and stability in the cscK problem. Ann. Sci. Éc. Norm. Supér. (4) **55**(6), 1529–1574 (2022)
21. Li, C., Tian, G., Wang, F.: On the Yau–Tian–Donaldson conjecture for singular Fano varieties. Commun. Pure Appl. Math. **74**(8), 1748–1800 (2021)
22. Li, C., Tian, G., Wang, F.: The uniform version of Yau–Tian–Donaldson conjecture for singular Fano varieties. Peking Math. J. **5**(2), 383–426 (2022)
23. Li, C., Wang, X., Xu, C.: On the proper moduli spaces of smoothable Kähler–Einstein Fano varieties. Duke Math. J. **168**(8), 1387–1459 (2019)
24. Liu, Y., Xu, C., Zhuang, Z.: Finite generation for valuations computing stability thresholds and applications to K-stability. Ann. Math. (2) **196**(2), 507–566 (2022)
25. Mabuchi, T.: K-energy maps integrating Futaki invariants. Tohoku Math. J. (2) **38**(4), 575–593 (1986)
26. Matsushima, Y.: Sur la structure du groupe d'homéomorphismes analytiques d'une certaine variété kählérienne. Nagoya Math. J. **11**, 145–150 (1957)
27. Ross, J., Witt Nyström, D.: Analytic test configurations and geodesic rays. J. Symplectic Geom. **12**(1), 125–169 (2014)
28. Rubinstein, Y.A., Tian, G., Zhang, K.: Basis divisors and balanced metrics. J. Reine Angew. Math. **778**, 171–218 (2021)
29. Tian, G.: K-stability and Kähler–Einstein metrics. Commun. Pure Appl. Math. **68**(7), 1085–1156 (2015)
30. Trusiani, A.: A relative Yau–Tian–Donaldson conjecture and stability thresholds. Adv. Math. **441**, Paper No. 109537, 95 pp. (2024)
31. Xu, C., Zhuang, Z.: On positivity of the CM line bundle on K-moduli spaces. Ann. Math. (2) **192**(3), 1005–1068 (2020)

32. Zhang, K.: A quantization proof of the uniform Yau–Tian–Donaldson conjecture. J. Eur. Math. Soc. **26**(12), 4763–4778 (2024)