A NOTE ON FILTRATIONS AND CHOW STABILITY

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Abstract. In this note we express the Chow weight using filtrations, which in turn is related to the slope at infinity of the quantized K-energy along Bergman geodesic rays.

Contents

1. Introduction

This note is motivated by the problem of finding constant scalar curvatuer Kähler metrics on polarized Kähler manifolds. The existence of such metrics is conjectured to be equivalent to certain algebro-geometric notion called *K-stability*, which goes back to Tian [27] and Donaldson [12]. For a comprehensive discussion of this subject, we refer the reader to Boucksom–Hisamoto–Jonsson [5]. In this note we will revisit a closely related and well-understood notion, i.e., *Chow stability*, which can be viewed as a "quantization" of K-stability. One aim of this note is to reformulate Chow stability using filtrations and it turns out that such consideration naturally relates Chow stability to the slope stability of certain quantized K-energy. Even though we believe that these contents are well-known to experts (see e.g., Székelyhid [26] for the same kind of treatment), we decide to revisit the details from a relatively more modern viewpoint. Anyhow, this treatment indeed has some consequences that might look a bit surprising at first glance; see for instance Corollary 1.11, which seems to be new. Another goal of this note is to study the relation between Chow stability and K-stability. This is of course an old topic. But as we shall see, using filtrations, one can extract some useful information that might serve as a new perspective to attack some open problems in this field. Some new stability thresholds characterizing Chow stability and K-stability are also introduced in the end.

1.1. **Conventions.** Let (X, L) be a polarized Kähler manifold of dimension n, where L is an ample line bundle. For simplicity we will further assume that mL is very ample and

$$H^i(X, mL) = 0 \text{ for all } i \geq 1$$

whenever $m \geq 1$. Set

$$R := R(X, L) := \bigoplus_{m\in\mathbb{N}} R_m,$$

where $R_m := H^0(X, mL)$. Also put

$$d_m := \dim R_m.$$

After replacing L by a suficiently divisible multiple we may (and will) assume that the subring

$$R^{(r)} := R(X, rL)$$

is generated in degree 1 for any $r \in \mathbb{N}_{>0}$. These harmless assumptions will make life a lot easier.

1.2. **Filtrations and test configurations.** We recall some standard facts about filtrations and test configurations. For more details, we refer to [30, 26] and [5, §2].

Definition 1.1. *(Filtration) We call $\mathcal{F}$ a filtration of R if for any $\lambda \in \mathbb{R}$ and $m \in \mathbb{N}$, there is subspace $\mathcal{F}^\lambda R_m$ of R_m satisfying*

1. *$\mathcal{F}^{\lambda'} R_m \supseteq \mathcal{F}^\lambda R_m$ for any $\lambda' \leq \lambda$;*
2. *$\mathcal{F}^\lambda R_m = \bigcap_{\lambda' < \lambda} \mathcal{F}^{\lambda'} R_m$;*
3. *$\mathcal{F}^{\lambda_1} R_{m_1} \cdot \mathcal{F}^{\lambda_2} R_{m_2} \subseteq \mathcal{F}^{\lambda_1+\lambda_2} R_{m_1+m_2}$ for any λ_1, λ_2 and $m_1, m_2 \in \mathbb{N}$;*
4. *$\mathcal{F}^\lambda R_m = R_m$ for $\lambda \leq 0$ and there exists $C > 0$ such that $\mathcal{F}^{Cm} R_m = \{0\}$ for any $m \in \mathbb{N}$.*

Given a filtration $\mathcal{F}$ of R, let

$$a_{m,i} := \inf\{\lambda \in \mathbb{R}_{\geq 0} | \mathrm{codim} \mathcal{F}^\lambda R_m \geq i\}$$

denote the *jumping numbers* of $\mathcal{F}$. We call $\mathcal{F}$ an $\mathbb{N}$-filtration if all $a_{m,i}$'s are nonnegative integers. Set

$$T_m(\mathcal{F}) := \frac{a_{m,d_m}}{m},\ S_m(\mathcal{F}) := \frac{1}{md_m} \sum_{i=1}^{d_m} a_{m,i} \text{ and } J_m(\mathcal{F}) := T_m(\mathcal{F}) - S_m(\mathcal{F}).$$

Put

$$T(\mathcal{F}) := \lim_{m\to\infty} T_m(\mathcal{F}),\ S(\mathcal{F}) := \lim_{m\to\infty} S_m(\mathcal{F}) \text{ and } J(\mathcal{F}) := \lim_{m\to\infty} J_m(\mathcal{F}).$$

(The existence of these limits are proved in [3].) One should think of $J(\mathcal{F})$ as certain norm of $\mathcal{F}$. We call $\mathcal{F}$ *trivial* at level m if $J_m(\mathcal{F}) = 0$. Note that in the literature the more general $\mathbb{R}$-filtrations and $\mathbb{Z}$-filtrations are investigated, but these can be easily reduced to our setting after an obvious translation of the parameter λ.

Definition 1.2 ([12])**.** *A test configuration $(\mathcal{X}, \mathcal{L})$ for (X, L) with exponent r consists of the following data:*

1. *a flat proper morphism π from a variety $\mathcal{X}$ to $\mathbb{C}$;*
2. *a $\mathbb{C}^*$-action on $\mathcal{X}$ lifting the canonical action on $\mathbb{C}$;*
3. *a $\mathbb{C}^*$-equivariant (relatively) very ample line bundle $\mathcal{L}$ on $\mathcal{X}$ such that $(\mathcal{X}_1, \mathcal{L}_1) \cong (X, rL)$.*

Definition 1.3. *An $\mathbb{N}$-filtration $\mathcal{F}$ of R is called finitely generated if the graded $\mathbb{C}[t]$-algebra (also called the Rees algebra)*

$$\bigoplus_{m\in\mathbb{N}} \left(\bigoplus_{\lambda\in\mathbb{Z}} t^{-\lambda} \mathcal{F}^\lambda R_m \right)$$

is finitely generated.

Given a finitely generated $\mathbb{N}$-filtration $\mathcal{F}$ of R, we may find a sufficiently divisible integer $r > 0$ such that

$$\bigoplus_{m\in\mathbb{N}} \left(\bigoplus_{\lambda\in\mathbb{Z}} t^{-\lambda} \mathcal{F}^\lambda R_{mr} \right)$$

is generated in degree 1. Put

$$\mathcal{X} := \mathrm{Proj}_{\mathbb{C}[t]} \bigoplus_{m\in\mathbb{N}} \left(\bigoplus_{\lambda\in\mathbb{Z}} t^{-\lambda} \mathcal{F}^\lambda R_{mr} \right) \text{ and } \mathcal{L} := \mathcal{O}_{\mathcal{X}}(1).$$

Then $(\mathcal{X}, \mathcal{L})$ gives rise to a test configuration of (X, L) with exponent r. Equivalently, $(\mathcal{X}, \mathcal{L})$ can be constructed explicitly as follows: Choose a basis $\{s_i\}_{1\leq i\leq d_r}$ of R_r such that

$$s_i \in \mathcal{F}^{a_{r,i}} R_r \text{ for each } 1 \leq i \leq d_r.$$

Then $\{s_i\}$ together with the 'weights' $\{a_{r,i}\}$ (i.e., the jumping numbers of $\mathcal{F}$ on R_r) induces a 1-parameter subgroup of $GL(d_r, \mathbb{C})$, which produces the flat family $\mathcal{X}$ inside $\mathbb{P}^{d_r-1} \times \mathbb{C}$ and $\mathcal{L}$ is the pullback of $\mathcal{O}_{\mathbb{P}^{d_r-1}\times\mathbb{C}}(1)$.

Given any integer $r > 0$, a basis $\{s_i\}_{1\leq r\leq d_r}$ of R_r and d_r non-negative integers $0 \leq \lambda_1 \leq ... \leq \lambda_{d_r}$, one can construct a finitely generated $\mathbb{N}$-filtration $\mathcal{F}$ of R as follows: Put for $\lambda \in \mathbb{R}$

$$\mathcal{F}^\lambda R_r := \mathrm{Span}_{\mathbb{C}}\{s_i | \lambda_i \geq \lambda\};$$

then one can extend this filtration of R_r to a fintely generated $\mathbb{N}$-filtration $\mathcal{F}$ of R using [4, Definition 3.18] (see also [26, §3.2]). A filtration thus obtained gives rive to a test configuration $(\mathcal{X}, \mathcal{L})$ with exponent

r, which clearly coincides with the one obtained by considering the 1-parameter subgroup of $GL(d_r, \mathbb{C})$ induced by the $\{s_i\}$ and the weights $\{\lambda_i\}$.

Conversely, any test configuration of (X, L) arises essentially in this way (see [22, Proposition 3.7]). More precisely, given a test configuration $(\mathcal{X}, \mathcal{L})$ of (X, L) with exponent r, then it is induced by a basis $\{s_i\}_{1\leq i\leq d_r}$ of R_r together with some weights $\{\lambda_i\}_{1\leq i\leq d_r}$. We may modify (this does not affect the stability notions to be introduced) the $\mathbb{C}^*$-action on $(\mathcal{X}, \mathcal{L})$ by a constant weight to make $\min_i\{\lambda_i\} = 0$. So each λ_i is a non-negative integer. Then the construction in the previous paragraph gives a finitely generated $\mathbb{N}$-filtration of R, which we will denote by $\mathcal{F}_{(\mathcal{X},\mathcal{L})}$. We also put

$$J(\mathcal{X}, \mathcal{L}) := J(\mathcal{F}_{(\mathcal{X},\mathcal{L})}).$$

This coincides with the J^{NA}-funtional of test configurations in the literature.

Definition 1.4 ([5])**.** *We call a test configuration $(\mathcal{X}, \mathcal{L})$ almost trivial if $J(\mathcal{X}, \mathcal{L}) = 0$. We call $(\mathcal{X}, \mathcal{L})$ trivial if it is a trivial product of (X, L) with $\mathbb{C}$.*

Geometrically, almost trivial means that, after normalization, $(\mathcal{X}, \mathcal{L})$ is a trivial.

1.3. **Chow stability and K-stability.** Next we attach a new invariant to filtrations. As we shall see this invariant has fruitful meanings when it comes to the stability notions of (X, L).

Definition 1.5. *Let $\mathcal{F}$ be a filtration of R. For $m \in \mathbb{N}_{>0}$, we call*

$$Chow_m(\mathcal{F}) := S(\mathcal{F}) - S_m(\mathcal{F})$$

the Chow invariant of $\mathcal{F}$ at level m.

Definition 1.6. *(Chow stability) We call the pair (X, L) Chow stable at level m if for any test configuration $(\mathcal{X}, \mathcal{L})$ with exponent m, one has*

$$Chow_m(\mathcal{F}_{(\mathcal{X},\mathcal{L})}) \geq 0,$$

and the equality holds only when $(\mathcal{X}, \mathcal{L})$ is trivial.

In the view of [30, 26], this definition coincides with the classical one in the literature.

Next, we recall the notion of K-stability, which was defined using the *generalized Futaki invariant*. This invariant first appeared in the work of Ding–Tian [10] and was later reformulated more algebraically by Donaldson [12].

Definition 1.7. *We call the pair (X, L) K-stable if any test configuration $(\mathcal{X}, \mathcal{L})$ with certain exponent r, the generalized Futaki invaraint*

$$Fut(\mathcal{X}, \mathcal{L}) := \lim_{m\to\infty} 2mr \cdot Chow_{mr}(\mathcal{F}_{(\mathcal{X},\mathcal{L})})$$

is always non-negative and it is zero only when $(\mathcal{X}, \mathcal{L})$ is almost trivial. We call the pair (X, L) uniformly K-stable if there exists $\varepsilon > 0$ such that

$$Fut(\mathcal{X}, \mathcal{L}) \geq \varepsilon J(\mathcal{X}, \mathcal{L})$$

for any test configuration $(\mathcal{X}, \mathcal{L})$ of (X, L).

1.4. **Main results.**

Theorem 1.8. *The pair (X, L) is Chow stable at level m if and only if there exists $\varepsilon > 0$ such that*

$$Chow_m(\mathcal{F}) \geq \varepsilon J_m(\mathcal{F})$$

for any filtration $\mathcal{F}$ of R.

Corollary 1.9. *Assume that (X, L) has discrete automorphism group, and that there exists a cscK metric in $c_1(L)$, then for any filtration $\mathcal{F}$ of R with $J(\mathcal{F}) > 0$ it holds that*

$$S(\mathcal{F}) > S_m(\mathcal{F}) \text{ for all } m \gg 1.$$

Proof. By Donaldson [11], for any $m \gg 1$, (X, L) is Chow stable at level m. Note also that $J_m(\mathcal{F}) > 0$ for $m \gg 1$. So the assertion follows from Theorem 1.8. □

Corollary 1.10. *If (X, L) is Chow stable at level m then $\delta_m(L) > \delta(L)$.*

Recall here that

$$\delta_m(L) := \inf_E \frac{A_X(E)}{S_m(E)} \text{ and } \delta_m(L) := \inf_E \frac{A_X(E)}{S(E)},$$

where E runs through all the prime divisors over X. Note that $A_X(E)$ is the log discrepancy of E and each E induces a filtration of R so $S(E)$ and $S_m(E)$ have obviously meanings.

Proof. There exists a prime divisor E over X satisfying

$$\delta_m(L) = \frac{A_X(E)}{S_m(E)}.$$

One also has $J_m(E) = T_m(E) - S_m(E) > 0$ as $|mL|$ is assumed to be base point free. Thus $S(E) > S_m(E)$ and hence

$$\delta_m(L) > \frac{A_X(E)}{S(E)} \geq \delta(L).$$

□

The next result might look a bit surprising at first glance. It would be interesting if one can find a purely algebraic proof of this.

Corollary 1.11. *Let X be a K-stable Fano manifold, then for all sufficiently large and divisible $m > 0$ one has*

$$\delta_m(-K_X) > \delta(-K_X).$$

Proof. By the Yau–Tian–Donaldson theorem [7, 28], there exists a Kähler–Einstein metric in $c_1(-K_X)$. Also note that X has discrete automorphism group. Then Donaldson's work [11] implies that $(X, -K_X)$ is Chow stable at sufficiently high levels. Choose a sufficiently divisible $r > 0$ such that the assumptions in §1.1 holds for $(X, -rK_X)$. Then we can apply the previous corollary to conclude. □

2. The proof

This section is devoted to the proof of Theorem 1.8. To start with, we recall the following well-known analytic characterization of Chow stability; see e.g., [19, 20, 21, 29] for several different proofs.

Theorem 2.1. *The pair (X, L) is Chow stable at level m if and only if (X, mL) admits a balanced embedding into $\mathbb{P}^{d_m-1}$.*

We will not recall the definition of balanced embedding as it is not needed in what follows. But one key point we shall recall is the following: As illustrated by Donaldson [13], the existence of balanced embeddings can be detected by certain functionals defined on the m-th Bergman space, which we now describe. Fix a positively curved smooth Hermitian metric h on L with $\omega := dd^c \log h > 0$ as its curvature form in $c_1(L)$. Put

$$\mathcal{H}_\omega := \left\{ \phi \in C^\infty(X, \mathbb{R}) \middle| \omega_\phi := \omega + dd^c\phi > 0 \right\}.$$

Let

$$E(\phi) := \frac{1}{(n+1)V} \sum_{i=0}^{n} \int_X \phi \omega^i \wedge \omega_\phi^{n-1}$$

denote the Monge–Amère energy for $\phi \in \mathcal{H}_\omega$, where $V := \int_X \omega^n$. The m-th Bergman space $\mathcal{B}_m$ is a subset in $\mathcal{H}_\omega$, whic is defined as

$$\mathcal{B}_m := \left\{ \phi = \frac{1}{m} \log \sum_{i=1}^{d_m} |s_i|^2_{h^m} \middle| \{s_i\} \text{ is a basis of } R_m \right\}.$$

Note that

$$H_m := \int_X h^m(\cdot, \cdot)\omega^n$$

defines a Hermitian inner product on the vector space R_m. Then simple linear algebra shows that, for any $\phi \in \mathcal{B}_m$, there exist an H_m-orthonormal basis $\{s_i\}_{1\leq i\leq d_m}$ of R_m and a set of real numbers $\{\lambda_i\}_{1\leq i\leq d_m}$ such that

$$\phi = \frac{1}{m} \log \sum_{i=1}^{d_m} e^{\lambda_i} |s_i|^2_{h^m}.$$

We may thus define the quantized Monge–Ampère energy as

$$E_m(\phi) := \frac{1}{md_m}\sum_{i=1}^{d_m}\lambda_i$$

and the quantized supremum as

$$T_m(\phi) := \frac{\max_{1\le i\le d_m}\{\lambda_i\}}{m}.$$

for $\phi \in \mathcal{B}_m$. Then by [13], together with the uniqueness of balanced embeddings [25] and the variational principle of [8], one has the following analytic criterion for the existence of balanced embeddings.

Theorem 2.2. *(X, mL) admits a balanced embedding into $\mathbb{P}^{d_m-1}$ if and only if the funcitonal $E - E_m$ is coercive on $\mathcal{B}_m$, namely, there exist $\varepsilon > 0$ and $C > 0$ such that*

$$E(\phi) - E_m(\phi) \ge \varepsilon(T_m(\phi) - E_m(\phi)) - C \text{ for all } \phi \in \mathcal{B}_m.$$

In this case the balanced embedding corresponds to the (unique) critical point of $E - E_m$ in $\mathcal{B}_m$.

Using the Bergman kernel asymptotic expansion, it is shown in [13] that

$$2m(E - E_m)$$

will converge to the classical K-energy of Mabuchi as $m \to \infty$. For this reason we call

$$K_m(\phi) := 2m(E(\phi) - E_m(\phi)) \text{ for } \phi \in \mathcal{B}_m$$

the *quantized K-energy at level m*.

The following result is clear.

Corollary 2.3. *The pair (X, L) is Chow stable at level m if and only if K_m is coercive on $\mathcal{B}_m$.*

Next we explain how do filtrations of R relate to the above variational picture. Given a filtration $\mathcal{F}$ of R, one can associate a 'geodesic ray' $\phi_t^{\mathcal{F}}$ to $\mathcal{F}$ using the construction in [23]. More precisely, choose an H_m-orthonormal basis $\{s_i\}$ of R_m such that

$$s_i \in \mathcal{F}^{a_{m,i}} R_m \text{ for all } 1 \le i \le d_m.$$

Namely $\{s_i\}$ is an H_m-orthonormal basis that is compatible with $\mathcal{F}$. Set for any $t \ge 0$

$$\phi_{m,t}^{\mathcal{F}} := \frac{1}{m}\log\sum_{i=1}^{d_m} e^{a_{m,i}t}|s_i|^2_{h^m}.$$

Then by definition,

$$S_m(\mathcal{F}) = \frac{E_m(\phi_{m,t}^{\mathcal{F}})}{t} \text{ and } T_m(\mathcal{F}) = \frac{T_m(\phi_{m,t}^{\mathcal{F}})}{t} \text{ for all } t > 0.$$

Now for $t \ge 0$ put

$$\phi_t^{\mathcal{F}} := \lim_{m\to\infty}\left(\sup_{k\ge m}\phi_{k,t}^{\mathcal{F}}\right)^*,$$

where $*$ denotes the upper semi-continuity regularization. By [31, (6.5)] one has

$$S(\mathcal{F}) = \frac{E(\phi_t^{\mathcal{F}})}{t} \text{ for all } t > 0.$$

As a consequence, we obtain that

$$Chow_m(\mathcal{F}) = S(\mathcal{F}) - S_m(\mathcal{F}) = \frac{E(\phi_t^{\mathcal{F}}) - E_m(\phi_{m,t}^{\mathcal{F}})}{t} \text{ for all } t > 0. \tag{2.1}$$

Lemma 2.4. *For any $\mathbb{N}$-filtration $\mathcal{F}$ of R, one has*

$$Chow_m(\mathcal{F}) \ge \lim_{t\to\infty}\frac{E(\phi_{m,t}^{\mathcal{F}}) - E_m(\phi_{m,t}^{\mathcal{F}})}{t}.$$

Proof. Denote by $\mathcal{F}_m$ the finitely generated $\mathbb{N}$-filtration of R generated by the filtration $\mathcal{F}|_{R_m}$ of R_m (see [4, Defnition 3.18]). So in particular,

$$S_m(\mathcal{F}) = S_m(\mathcal{F}_m) \text{ and } \phi^{\mathcal{F}}_{m,t} = \phi^{\mathcal{F}_m}_{m,t}.$$

On the other hand, for any $k \in \mathbb{N}_{>0}$ one has

$$S_{mk}(\mathcal{F}) \geq S_{mk}(\mathcal{F}_m)$$

So sending $k \to \infty$ we conclude that

$$S(\mathcal{F}) \geq S(\mathcal{F}_m).$$

Now let $(\mathcal{X}, \mathcal{L})$ be the test configuration of (X, L) induced by $\mathcal{F}_m$ and consider the associated geodesic ray $\phi^{\mathcal{F}_m}_t$. This geodesic ray coincides with the one obtained by Berman [1, Proposition 2.7] and the estimate in *loc. cit.* shows that

$$|\phi^{\mathcal{F}_m}_t - \phi^{\mathcal{F}_m}_{m,t}| < C$$

for some uniform $C > 0$ (note that $\phi^{\mathcal{F}_m}_{m,t}$ is denoted by ϕ_{FS} in [1]). Therefore

$$S(\mathcal{F}) - S_m(\mathcal{F}) \geq S(\mathcal{F}_m) - S_m(\mathcal{F}_m) = \lim_{t\to\infty} \frac{E(\phi^{\mathcal{F}_m}_t) - E_m(\phi^{\mathcal{F}_m}_{m,t})}{t} = \lim_{t\to\infty} \frac{E(\phi^{\mathcal{F}}_{m,t}) - E_m(\phi^{\mathcal{F}}_{m,t})}{t},$$

as desired. □

Remark 2.5. It is clear from the above proof that the equality

$$Chow_m(\mathcal{F}) = \lim_{t\to\infty} \frac{E(\phi^{\mathcal{F}}_{m,t}) - E_m(\phi^{\mathcal{F}}_{m,t})}{t}. \tag{2.2}$$

holds when $\mathcal{F} = \mathcal{F}_{(\mathcal{X},\mathcal{L})}$ is induced by a test configuration of (X, L) with exponent m. This is probably well-known and is a quantized version of Chi Li's result [16, Theorem 1.7.2].

We are now ready to prove our main result.

Proof of Theorem 1.8. The 'if' part follows immediately from the definition.

For the 'only if' part, we first treat the case where $\mathcal{F}$ is an $\mathbb{N}$-filtration of R. By the previous lemma,

$$Chow_m(\mathcal{F}) \geq \lim_{t\to\infty} \frac{E(\phi^{\mathcal{F}}_{m,t}) - E_m(\phi^{\mathcal{F}}_{m,t})}{t}.$$

On the other hand, (X, L) being Chow stability at level m implies that, for some $\varepsilon_0 > 0$ and $C_0 > 0$,

$$E(\phi^{\mathcal{F}}_{m,t}) - E_m(\phi^{\mathcal{F}}_{m,t}) \geq \varepsilon_0\big(T_m(\phi^{\mathcal{F}}_{m,t}) - E_m(\phi^{\mathcal{F}}_{m,t})\big) - C_0.$$

Now using the fact that

$$J_m(\mathcal{F}) = \frac{T_m(\phi^{\mathcal{F}}_{m,t}) - E_m(\phi^{\mathcal{F}}_{m,t})}{t} \text{ for any } t > 0,$$

we arrive at

$$S(\mathcal{F}) - S_m(\mathcal{F}) \geq \varepsilon_0 J_m(\mathcal{F})$$

for any $\mathbb{N}$-filtration of R. Next we claim that the same estimate holds for all filtrations of R. To see this, consider the quantity

$$U(\mathcal{F}) := \frac{S(\mathcal{F}) - S_m(\mathcal{F})}{J_m(\mathcal{F})}$$

for filtrations $\mathcal{F}$ of R with $J_m(\mathcal{F}) > 0$ (when $J_m(\mathcal{F}) = 0$ the desired estimate trivially holds). We wish to show that

$$\inf_{\mathcal{F}} U(\mathcal{F}) \geq \varepsilon_0.$$

Obviously, $U(\mathcal{F})$ is invariant under scaling and translation of $\mathcal{F}$. So it sufficies to consider those $\mathcal{F}$ with $J_m(\mathcal{F}) \geq 1$. Given such an $\mathcal{F}$, let

$$0 \leq a_{m,1} \leq ... \leq a_{m,d_m}$$

be its (possibly irrational) jumping numbers at level m. Using Dirichlet's approximation theorem, for any small $\varepsilon > 0$, we can find an integer $p > 0$ and $(q_1, ..., q_{d_m}) \in \mathbb{N}^{d_m}$ such that

$$|pa_{m,i} - q_i| < \varepsilon \text{ for any } 1 \leq i \leq d_m.$$

We now scale $\mathcal{F}$ by factor q and then translate it by constant ε. By abuse of notation we still denote the resulting filtration by $\mathcal{F}$ (note that $U(\mathcal{F})$ is unchanged). The jumping numbers of $\mathcal{F}$ at level m then becomes $\lambda_i := pa_{m,i} + \varepsilon$ and $J_m(\mathcal{F}) \geq p \geq 1$. Note also that

$$q_i < \lambda_i < q_i + 2\varepsilon \text{ for any } 1 \leq i \leq d_m.$$

Now consider the $\mathbb{N}$-filtration $\mathcal{F}_{\mathbb{N}}$ induced by $\mathcal{F}$, which is given by (see also [3, §2.7])

$$\mathcal{F}_{\mathbb{N}}^{\lambda} R_k := \mathcal{F}^{\lceil \lambda \rceil} R_k \text{ for } \lambda \in \mathbb{R} \text{ and } k \in \mathbb{N}.$$

Then the jumping numbers of $\mathcal{F}_{\mathbb{N}}$ at level m are $q_1, ..., q_{d_m}$. So one easily sees that

$$S(\mathcal{F}) = S(\mathcal{F}_{\mathbb{N}}),\ S_m(\mathcal{F}) - 2\varepsilon < S_m(\mathcal{F}_{\mathbb{N}}) < S_m(\mathcal{F}),$$

and

$$J_m(\mathcal{F}_{\mathbb{N}}) > J_m(\mathcal{F}) - 2\varepsilon \geq 1 - 2\varepsilon.$$

Therefore,

$$U(\mathcal{F}) \geq \frac{S(\mathcal{F}_{\mathbb{N}}) - S_m(\mathcal{F}_{\mathbb{N}}) - 2\varepsilon}{J_m(\mathcal{F}_{\mathbb{N}}) + 2\varepsilon} \geq \frac{1}{1 + \frac{2\varepsilon}{1-2\varepsilon}} \cdot \left(U(\mathcal{F}_{\mathbb{N}}) - \frac{2\varepsilon}{1-2\varepsilon} \right) \geq \frac{1}{1 + \frac{2\varepsilon}{1-2\varepsilon}} \cdot \left(\varepsilon_0 - \frac{2\varepsilon}{1-2\varepsilon} \right).$$

Now sending $\varepsilon \to 0$ we complete the proof. □

Using the above argument, for any filtration $\mathcal{F}$ of R, after scaling and translation, the Bergman geodesic ray $\phi^{\mathcal{F}}_{m,t}$ can be approximated by a Bergman geodesic ray induced from an $\mathbb{N}$-filtration as close as we want. As a consequence, we can strengthen Lemma 2.4 as follows:

Lemma 2.6. *For any filtration $\mathcal{F}$ of R, one has*

$$Chow_m(\mathcal{F}) \geq \lim_{t \to \infty} \frac{E(\phi^{\mathcal{F}}_{m,t}) - E_m(\phi^{\mathcal{F}}_{m,t})}{t}.$$

Corollary 2.7. *For any filrtation $\mathcal{F}$ of R with $J_m(\mathcal{F}) > 0$, one has*

$$\frac{S(\mathcal{F}) - S_m(\mathcal{F})}{J_m(\mathcal{F})} \geq \lim_{t \to \infty} \frac{E(\phi^{\mathcal{F}}_{m,t}) - E_m(\phi^{\mathcal{F}}_{m,t})}{T_m(\phi^{\mathcal{F}}_{m,t}) - E_m(\phi^{\mathcal{F}}_{m,t})}.$$

Proof. We conclude using the identity $T_m(\phi^{\mathcal{F}}_{m,t}) - E_m(\phi^{\mathcal{F}}_{m,t}) = J_m(\mathcal{F})t$. □

3. Further discussions

In this part we make some general discussions and propose several questions/conjectures for future research.

3.1. **Limiting behavior.** For any filtration $\mathcal{F}$ of R, it is an interesting question to see if the limit

$$\lim_{m \to \infty} 2m \cdot Chow_m(\mathcal{F}) = \lim_{m \to \infty} 2m(S(\mathcal{F}) - S_m(\mathcal{F}))$$

exists or not. The answer seems to be negative as it turns out to be related to the problem of counting lattice points in certain filtrated Okounkov body (a problem of this sort is already rather non-trivial even for rational polytopes; see e.g. [18]). When $\mathcal{F}$ is a finitely generated $\mathbb{N}$-filtration of R, the following result is easy to show.

Lemma 3.1. *When $\mathcal{F}$ is a finitely generated $\mathbb{N}$-filtration of R, the limit*

$$\lim_{m \to \infty} 2m(S(\mathcal{F}) - S_m(\mathcal{F}))$$

can possibly take finitely many different values, depending on the divisibility of m as it goes to infinity. While for any sufficiently divisible $r \in \mathbb{N}_{>0}$, one has

$$Fut(\mathcal{X}, \mathcal{L}) = \lim_{m \to \infty} 2mr(S(\mathcal{F}) - S_{mr}(\mathcal{F})),$$

where $(\mathcal{X}, \mathcal{L})$ is some test configuration of (X, L) induced by $\mathcal{F}$.

Proof. Since $\mathcal{F}$ is finitely generated, it implies that the bi-graded ring

$$\bigoplus_{m\in\mathbb{N}}\left(\bigoplus_{\lambda\in\mathbb{N}}\mathcal{F}^\lambda R_m\right)$$

is finitely generated over $\mathbb{C}$. It then follows from the theory of Hilbert functions that

$$\sum_{\lambda\in\mathbb{N}}\dim\mathcal{F}^\lambda R_m=Q(m)$$

for some *quasi-polynomial* Q whenever m is large enough. This means that Q is a polynomial in m, whose coefficients depend on m periodically. On the other hand, by Riemann–Roch we also have for $m\gg 1$

$$d_m=P(m)$$

for some polynomial of degree n. As $\mathcal{F}$ is assumed to be linearly bounded, Q has degree at most $n+1$. So we can write

$$Q(m)=b_{n+1}(m)m^{n+1}+b_m(m)m^n+...+b_0(m),$$

where each $b_i:\mathbb{N}\to\mathbb{Q}$ is a periodic function. Now using the fact

$$S_m(\mathcal{F})=\frac{Q(m)-d_m}{md_m}=\frac{Q(m)-P(m)}{mP(m)},$$

we see that the leading coefficient b_{n+1} of Q is a constant depending on $S(\mathcal{F})$ and $\mathrm{vol}(L)$. However starting from the second leading coefficient, each $b_i(m)$ can possibly take finitely many different values. It then follows that the limit $\lim_{m\to\infty}2m(S(\mathcal{F})-S_m(\mathcal{F}))$ can only take finitely many different values, depending on the divisibility of m.

But for any sufficiently divisible $r\in\mathbb{N}_{>0}$, the bi-graded ring

$$\bigoplus_{m\in\mathbb{N}}\left(\bigoplus_{\lambda\in\mathbb{N}}\mathcal{F}^\lambda R_{mr}\right)$$

is generated in degree 1. Then the Hilbert function $Q(mr)$ becomes a polynomial in m whenever m is sufficiently large. In this case one easily sees that $\lim_{m\to\infty}2mr(S(\mathcal{F})-S_{mr}(\mathcal{F}))$ exists and coincides with the generalized Futaki invariant of any test configuration associated to $\mathcal{F}$. □

Corollary 3.2. *Let $\mathcal{F}$ be a finitely generated $\mathbb{N}$-filtration of R. Then for any sufficiently divisible $r\in\mathbb{N}_{>0}$, one has*

$$\lim_{t\to\infty}\frac{K(\phi_t^{\mathcal{F}})}{t}\le\lim_{m\to\infty}\left(\lim_{t\to\infty}\frac{K_{mr}(\phi^{\mathcal{F}}_{mr,t}))}{t}\right),$$

where

$$K(\phi):=\frac{1}{V}\log\int_X\frac{\omega_\phi^n}{\omega^n}\omega_\phi^n+n\frac{(-K_X)\cdot L^{n-1}}{L^n}E(\phi)-\frac{1}{V}\int_X\phi\,\mathrm{Ric}(\omega)\wedge\sum_{i=0}^{n-1}\omega^i\wedge\omega_\phi^{n-1-i}$$

denotes the K-energy and $K_m:=2m(E-E_m)$ is the quantized K-energy. The equal holds if the test configuration induced by $\mathcal{F}$ has reduced central fiber.

Proof. Fix $r_0\in\mathbb{N}_{>0}$ such that for any multiple r of r_0, the $\mathbb{C}[t]$-algebra

$$\bigoplus_{m\in\mathbb{N}}\left(\bigoplus_{\lambda\in\mathbb{Z}}t^{-\lambda}\mathcal{F}^\lambda R_{mr}\right)$$

is generated in degree 1. Then as explained in Remark 2.5, one has

$$2mr(S(\mathcal{F})-S_{mr}(\mathcal{F}))=\lim_{t\to\infty}\frac{K_{mr}(\phi^{\mathcal{F}}_{mr,t}))}{t}\text{ for all }m\in\mathbb{N}_{>0}.$$

Let $(\mathcal{X},\mathcal{L})$ be the test configuration induced by $\mathcal{F}$ with exponent r_0. Applying the previous lemma, for any r divisible by r_0, we deduce that

$$Fut(\mathcal{X},\mathcal{L})=\lim_{m\to\infty}\left(\lim_{t\to\infty}\frac{K_{mr}(\phi^{\mathcal{F}}_{mr,t}))}{t}\right).$$

On the other hand, by Li [16, Theorem 1.7.2], one has

$$\lim_{t\to\infty}\frac{K(\phi_t^{\mathcal{F}})}{t}\le Fut(\mathcal{X},\mathcal{L}),$$

and the equality holds if the central fiber $\mathcal{X}_0$ is reduced. Thus we get the assertion. □

When $\mathcal{F}$ is a finitely generated $\mathbb{N}$-filtration, it follows from Lemma 3.1 that

$$Fut(\mathcal{X},\mathcal{L}) \geq \liminf_{m\to\infty} 2m(S(\mathcal{F}) - S_m(\mathcal{F})), \tag{3.1}$$

where $(\mathcal{X},\mathcal{L})$ is the test configuration of (X,L) induced by $\mathcal{F}$. A bit more careful analysis can probably show that the above inequality is actually an equality when the central fiber $\mathcal{X}_0$ is reduced (this is possibly not hard to show but at the moment it is not clear to the author).

For a general filtration $\mathcal{F}$ of R, Lemma 2.6 implies that

$$\liminf_{m\to\infty} 2m(S(\mathcal{F}) - S_m(\mathcal{F})) \geq \liminf_{m\to\infty}\left(\lim_{t\to\infty}\frac{2m\big(E(\phi^{\mathcal{F}}_{m,t}) - E_m(\phi^{\mathcal{F}}_{m,t})\big)}{t}\right). \tag{3.2}$$

Note that the right hand side is also investigated in [26], which is denoted by $Chow_\infty(\mathcal{F})$ in *loc. cit.*

Conjecture 3.3. *One has*

(1) $\liminf_{m\to\infty} 2m(S(\mathcal{F}) - S_m(\mathcal{F}))$ *is equal to the non-Archimedean Mabuchi energy in* [5, §7.3]*;*

(2) $\liminf_{m\to\infty}\left(\lim_{t\to\infty}\frac{2m\left(E(\phi^{\mathcal{F}}_{m,t}) - E_m(\phi^{\mathcal{F}}_{m,t})\right)}{t}\right)$ *is equal to the slope at infinity of K along $\phi^{\mathcal{F}}_t$.*

If this conjecture holds then combining (3.2) with [16, Theorem 1.7.1] one can deduce that [16, Conjecture 1.6] holds for those maximal geodesic rays induced by filtrations. Probably this conjecture is too good to be true, but at least it is promising for those filtrations that are induced by *models* of (X,L) (in the sense of [16]), in which case we actually expect that $\liminf_{m\to\infty} 2m(S(\mathcal{F}) - S_m(\mathcal{F}))$ is given by the intersection formula in [17, Theorem 1.1].

Remark 3.4. Given a filtration $\mathcal{F}$ of R, a natural idea is to approximate $\mathcal{F}$ by a sequence of finitely generated filtrations (as in [26]). Then to derive useful information, one key point is to show that the Hilbert function associated to these filtrations stay in a "bounded family". This seems to be related to the *Fujita approximation conjecture* of Li [17].

3.2. **Some new stability thresholds.** The following estimate is building on [2, Lemma 7.7] and [24, Lemma 5.2].

Proposition 3.5. *There exists $C_0 > 0$ such that for any filtration $\mathcal{F}$ of R with $J(\mathcal{F}) > 0$, one has*

$$\liminf_{m\to\infty}\frac{2m(S(\mathcal{F}) - S_m(\mathcal{F}))}{J_m(\mathcal{F})} > -C_0.$$

Proof. Given any $m \gg 1$, we consider the Bergman geodesic ray $\phi^{\mathcal{F}}_{m,t}$. By definition, there exists an H_m-orthonormal basis $\{s_i\}_{1\leq i\leq d_m}$ of R_m such that

$$\phi^{\mathcal{F}}_{m,t} = \frac{1}{m}\log\sum_{i=1}^{d_m} e^{a_{m,i}t}|s_i|^2_{h^m},$$

where $a_{m,i}$'s are the jumping numbers of $\mathcal{F}$. Then using the convexity of E along Bergman geodesics, one has for any $t > 0$

$$\begin{aligned}\frac{E(\phi^{\mathcal{F}}_{m,t}) - E(\phi^{\mathcal{F}}_{m,0})}{t} &\geq \frac{d}{dt}\Big|_{t=0} E(\phi^{\mathcal{F}}_{m,t}) = \frac{1}{mV}\int_X \frac{\sum_{i=1}^{d_m} a_{m,i}|s_i|^2_{h^m}}{\sum_{i=1}^{d_m}|s_i|^2_{h^m}}\omega^n_{\phi^{\mathcal{F}}_{m,0}}\\ &= \frac{1}{mV}\int_X \frac{\sum_{i=1}^{d_m}(a_{m,i} - a_{m,d_m})|s_i|^2_{h^m}}{\sum_{i=1}^{d_m}|s_i|^2_{h^m}}\omega^n_{\phi^{\mathcal{F}}_{m,0}} + \frac{a_{m,d_m}}{m}.\end{aligned}$$

Observe that $\phi^{\mathcal{F}}_{m,0} = \frac{1}{m}\log\sum_{i=1}^{d_m}|s_i|^2_{h^m}$. So by the standard Bergman kernel expansion, one can find $C_0 > 0$ (independent of $\mathcal{F}$) such that

$$\frac{d_m\omega^n_{\phi^{\mathcal{F}}_{m,0}}}{V\cdot\sum_{i=1}^{d_m}|s_i|^2_{h^m}} \leq (1 + \frac{C_0}{m})\omega^n$$

whenever m is sufficiently large. This implies that

$$\begin{aligned}\frac{E(\phi^{\mathcal{F}}_{m,t})-E(\phi^{\mathcal{F}}_{m,0})}{t} &\geq \frac{1+C_0/m}{md_m}\int_X\sum_{i=1}^{d_m}(a_{m,i}-a_{m,d_m})|s_i|^2_{h^m}\omega^n+\frac{a_{m,d_m}}{m}\\ &=\frac{1+C_0/m}{md_m}\sum_{i=1}^{d_m}a_{m,i}-\frac{C_0}{m}\cdot\frac{a_{m,d_m}}{m}\\ &=\frac{1+C_0/m}{t}E_m(\phi^{\mathcal{F}}_{m,t})-\frac{C_0}{m}\cdot\frac{T_m(\phi^{\mathcal{F}}_{m,t})}{t}.\end{aligned}$$

Rearranging, we obtain that

$$\frac{E(\phi^{\mathcal{F}}_{m,t})-E_m(\phi^{\mathcal{F}}_{m,t})}{T_m(\phi^{\mathcal{F}}_{m,t})-E_m(\phi^{\mathcal{F}}_{m,t})}\geq-\frac{C_0}{m}+\frac{E(\phi^{\mathcal{F}}_{m,0})}{T_m(\phi^{\mathcal{F}}_{m,t})-E_m(\phi^{\mathcal{F}}_{m,t})}\text{ for any } t>0.$$

Now letting $t\to\infty$, we arrive at

$$\lim_{t\to\infty}\frac{E(\phi^{\mathcal{F}}_{m,t})-E_m(\phi^{\mathcal{F}}_{m,t})}{T_m(\phi^{\mathcal{F}}_{m,t})-E_m(\phi^{\mathcal{F}}_{m,t})}\geq-\frac{C_0}{m},$$

which holds for all $m\gg 1$. Then using Corollary 2.7, we conclude. □

Therefore, for those $\mathcal{F}$ with $J(\mathcal{F})>0$, it makes sense to consider the following quantity:

$$\gamma(\mathcal{F}):=\liminf_{m\to\infty}\frac{2m(S(\mathcal{F})-S_m(\mathcal{F}))}{J(\mathcal{F})}.$$

Put

$$\gamma(L):=\inf_{\mathcal{F}}\gamma(\mathcal{F}), \tag{3.3}$$

where $\mathcal{F}$ runs through all the filtrations with $J(\mathcal{F})>0$. Note that by Proposition 3.5,

$$\gamma(L)>-\infty.$$

Similarly one can define a quantized version of γ by putting

$$\gamma_m(L):=\inf_{\mathcal{F}}\frac{2m(S(\mathcal{F})-S_m(\mathcal{F}))}{J_m(\mathcal{F})}.$$

where $\mathcal{F}$ satisfies $J_m(\mathcal{F})>0$. By the proof of Theorem 1.8, in the above inf, it is enough to only consider $\mathbb{N}$-filtrations.

The following result is clear.

Proposition 3.6. *One has*

1. *(X,L) is Chow stable at level m if and only if $\gamma_m(L)>0$;*
2. *(X,L) is K-stable if and only if $\gamma(\mathcal{F})>0$ for any finitely generated $\mathbb{N}$-filtration $\mathcal{F}$ with $J(\mathcal{F})>0$;*
3. *(X,L) is uniformly K-stable if $\gamma(L)>0$;*
4. *(X,L) is uniformly K-stable if $\liminf\gamma_m(L)>0$.*

When $\mathcal{F}=\mathcal{F}_v$ is induced by some valuation $v\in\mathrm{Val}_X$, one has (cf. [6])

$$\frac{S(\mathcal{F}_v)}{n}\leq J(\mathcal{F}_v)\leq nS(\mathcal{F}_v).$$

For this reason one can also consider

$$\eta(v):=\liminf_{m\to\infty}\frac{2m(S(\mathcal{F}_v)-S_m(\mathcal{F}_v))}{S(\mathcal{F}_v)}.$$

Remark 3.7. When X is Fano and $L=-K_X$, then for any *dreamy* divisor E over X, one has (cf. [15])

$$\eta(\mathrm{ord}_E)=\frac{A_X(E)}{S(E)}-1.$$

In this case the liminf in the definition of η is actually a limit, since one has an explicit asymptotic expansion of S_m using K. Fujita's asymptotic Riemann-Roch [14, Proposition 4.1]. See also [9] for the case of general polarizations.

We end this note by asking the following

Question 3.1. *Under what conditions is the inf in (3.3) computed by some finitely generated $\mathbb{N}$-filtration?*

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