The greatest Ricci lower bound and stability threshold

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Abstract In this note, we discuss the relation between the greatest Ricci lower bound and δ-invariant on Fano manifolds.

1. Motivation

Throughout this note, X will be an n-dimensional Fano manifold. An interesting problem in Kähler geometry is to find a canonical metric on X. Over the past a few decades, various energy functionals and invariants have been introduced to tackle this problem. We refer the reader to the recent survey [16] for a nice introduction. In this note, we will be mainly interested in two invariants defined on X. One is known as the greatest Ricci lower bound, and the other is known as the δ-invariant (also referred to as the K-stability threshold). The purpose of this note is to study the relation between them.

2. Background

We recall the following definitions.

Definition 1.1 ([17, 20]). *We define the greatest Ricci lower bound $R(X)$ to be*

$$R(X) := \sup\{\lambda > 0 \mid \exists\ \omega \in 2\pi c_1(X) \text{ such that } Ric(\omega) > \lambda\omega\ \}.$$

Definition 1.2 ([2, 14]). *Let L be an ample $\mathbb{Q}$-line bundle on X. For any sufficiently large and divisible integer k, we consider a basis $s_1, \cdots s_{d_k}$ of $H^0(X, kL)$,* *where* $d_k = h^0(X, kL)$. *We can associate a* $\mathbb{Q}$*-divisor* $D \sim_{\mathbb{Q}} L$ *to this basis by*

$$D := \frac{1}{kd_k} \sum_{i=1}^{d_k} \{s_i = 0\}.$$

Any D *obtained in this way is called a* k*-basis type divisor of* L*. We put*

$\delta_k(L) := \sup\{c > 0 \mid (X, cD)$ *is log canonical for any* k*-basis type divisor* D *of* $L\}$.

Then we define $\delta(L)$ *by*

$$\delta(L) := \limsup_k \delta_k(L).$$

If $L = -K_X$*, then we simply write*

$$\delta(X) := \delta(-K_X),$$

which is called the δ*-invariant of* X*.*

Remark 1. *The thresholds* $R(X)$ *and* $\delta(X)$ *are also closely related to the alpha invariant* $\alpha(X)$ *introduced by Tian [19]. For instance, we have*

$$R(X) \geqslant \min\left\{\frac{n+1}{n}\alpha(X), 1\right\},$$

which can be derived using the continuity method; see [19]. Moreover, we have (see [2, Theorem A] for a more general result)

$$\frac{n+1}{n}\alpha(X) \leqslant \delta(X) \leqslant (n+1)\alpha(X).$$

It turns out that $R(X)$ and $\delta(X)$ are quite useful to test K-(semi)stability of X (for the definition of K-(semi)stability, we refer the reader to [1]). In particular, we recall the following result.

Theorem 2 ([2, 10, 14])**.** *The following are equivalent.*

1. X *is K-semistable;*
2. $R(X) = 1$*;*
3. $\delta(X) \geqslant 1$*.*

3. Main result

Roughly speaking, $R(X)$ measures how far X is from being a Kähler-Einstein (KE) manifold. So it is always an interesting problem to find the value of $R(X)$ since it plays an important role in the study of KE problems. We refer the reader to [9, 11, 17, 18, 20] for more details in this direction. The main result of this note is the following result.

Theorem 3 ([5]). *Let X be a Fano manifold. Then we have*

$$R(X) = \min\{\delta(X), 1\}.$$

This result was proved in the toric case by Blum-Jonsson [2, Corollary 7.19]. So it is reasonable to believe that the same result holds in the general setting. The purpose of this note is to give a short proof of Theorem 3 relying on some recent developments in the literature.

4. The proof

To prove Theorem 3, one needs to use Kähler-Einstein edge (KEE) metric and its corresponding thresholds as well. So we recall the following two definitions.

Definition 1.3 ([10]). *Suppose that $\Delta \in |-mK_X|$ is a smooth divisor, where m is a positive integer. We define*

$$R(X, \Delta/m) := \sup\{\lambda > 0 \mid \exists \textit{ KEE metric } \omega \in 2\pi c_1(X) \textit{ s.t. } Ric(\omega) = \lambda\omega + 2\pi(1-\lambda)[\Delta]/m\}$$

Definition 1.4 ([7]). *Suppose that $\Delta \in |-mK_X|$ is a smooth divisor, where m is a positive integer. Let $\lambda \in (0, 1]$ be a rational number. Then $-K_X - \frac{1-\lambda}{m}\Delta \sim_{\mathbb{Q}} -\lambda K_X$ is an ample $\mathbb{Q}$-line bundle. We define*

$$\delta_k(X, \frac{1-\lambda}{m}\Delta) := \sup\left\{ c > 0 \;\middle|\; \begin{array}{l} \textit{the log pair } \left(X, \frac{1-\lambda}{m}\Delta + cD\right) \textit{ is log canonical} \\ \textit{for any } k\textit{-basis type divisor } D \textit{ of } -\lambda K_X \end{array} \right\}.$$

Moreover, we define

$$\delta(X, \frac{1-\lambda}{m}\Delta) := \limsup_k \delta_k(X, \frac{1-\lambda}{m}\Delta),$$

which is called the δ-invariant of the log Fano pair $(X, \frac{1-\lambda}{m}\Delta)$.

Remark 4. *For more information about KE and KEE metrics, we refer to [15].*

Remark 5. *By Bertini's theorem, for $m \gg 1$, any general divisor $\Delta \in |-mK_X|$ is smooth.*

Remark 6. *$R(X)$ and $R(X,\Delta/m)$ can be related as follows:*

$$R(X)\frac{m-1}{m-R(X)} \leqslant R(X,\Delta/m) \leqslant R(X). \tag{1.5}$$

See [18] for a proof (see also [10]). In particular, $\lim R(X,\Delta/m) = R(X)$.

Remark 7. *The thresholds $R(X,\Delta/m)$ and $\delta(X,\frac{1-\lambda}{m}\Delta)$ are the counterparts of $R(X)$ and $\delta(X)$ in the log setting. They can be used to test the existence of KEE metrics and log K-(semi)stability (see e.g. [7, 8]).*

Remark 8. *If follows immediately from the definition that*

$$\delta(-\lambda K_X) \geqslant \delta(X,\frac{1-\lambda}{m}\Delta). \tag{1.6}$$

With a little more effort, one can actually prove the following

Lemma 9. *Fix a rational number $\lambda \in (0,1]$. For each $m \gg 1$ pick a smooth divisor $\Delta \in |-mK_X|$ and put $B_m := \frac{1-\lambda}{m}\Delta$. Then we have*

$$\lim_{m\to\infty} \delta(X,B_m) = \delta(-\lambda K_X).$$

Proof. Fix any small $\epsilon > 0$. If suffices to show that, for any $m \gg 1$ and sufficiently divisible $k \gg 1$, we have

$$\delta_k(-\lambda K_X) \geqslant \delta_k(X,B_m) \geqslant (1-\epsilon)\delta_k(-\lambda K_X).$$

The first inequality $\delta_k(-\lambda K_X) \geqslant \delta_k(X,B_m)$ follows clearly from Definition 1.2 and Definition 1.4. So it remains to prove the second inequality. For this purpose, we let D be any k-basis type divisor of $-\lambda K_X$. Pick any $c > 0$ such that the log pair (X,cD) is log canonical. If then suffices to show that the log pair $(X,B_m+(1-\epsilon)cD)$ is log canonical as well.

Now notice that, the log pair (X,Δ) is log canonical since $\Delta \in |-mK_X|$ is a *smooth* divisor. Then we can apply a trick from [4] to show the log canonicity of $(X,B_m+(1-\epsilon)cD)$. Indeed, suppose that the log pair $(X,B_m+(1-\epsilon)cD)$ is not log canonical, then [4, Remark 2.1] implies that the log pair $(X,\frac{(1-\epsilon)c}{1-(1-\lambda)/m}D)$ is not log canonical as well. If we pick $m \geqslant \frac{1-\lambda}{\epsilon}$, the log pair (X,cD) is then not log canonical, contradicting our choice of c. □

Now we are ready to prove Theorem 3.

Proof of Theorem 3. Using Theorem 2, it is enough to assume that X is not K-semistable. So $R(X) \in (0,1)$. Our goal is to show that $\delta(X) = R(X)$. For simplicity we may also assume that $R(X) \in \mathbb{Q}$. Then we consider the ample $\mathbb{Q}$-line bundle $-R(X)K_X$. By Definition 1.2, it suffices to show that

$$\delta(-R(X)K_X) = 1.$$

First, we show that $\delta(-R(X)K_X) \geqslant 1$. For this purpose, we pick any rational number $\lambda \in (0, R(X))$. Then let m be a sufficiently large integer and pick a smooth divisor $\Delta \in |-mK_X|$. By (1.5), we may assume that

$$\lambda < R(X, \Delta/m).$$

Then by Definition 1.3 and [8, Theorem 1.1], we can find a KEE metric $\omega \in 2\pi c_1(X)$ such that

$$Ric(\omega) = \lambda\omega + 2\pi(1-\lambda)[\Delta]/m.$$

So the log pair $(X, \frac{\lambda-1}{m}\Delta)$ is log K-semistable (see [8, Corollary 1.12]). Thus by [7, Corollary 4.8], we have

$$\delta(X, \frac{\lambda-1}{m}\Delta) \geqslant 1.$$

Hence by (1.6), we have $\delta(-\lambda K_X) \geqslant 1$. Letting $\lambda \to R(X)$, we get

$$\delta(-R(X)K_X) \geqslant 1.$$

So it remains to show that $\delta(-R(X)K_X) \leqslant 1$. We argue by contradiction. Suppose that $\delta(-R(X)K_X) > 1$. Then we may pick a sufficiently small rational number $\epsilon > 0$ such that $R(X) + \epsilon \leqslant 1$ (recall that $R(X) < 1$) and

$$\delta\big(-(R(X)+\epsilon)K_X\big) > 1.$$

Then Lemma 9 implies that, for any $m \gg 1$ and any smooth divisor $\Delta \in |-mK_X|$, we have

$$\delta(X, \frac{1-(R(X)+\epsilon)}{m}\Delta) > 1.$$

Then by [3, Corollary 2.11], the log pair $\big(X, \frac{1-(R(X)+\epsilon)}{m}\Delta\big)$ is uniformly log K-stable. So it follows from [6, 21] (see also [22]) that, there exists a KEE metric associated to this pair. Thus we have $R(X, \Delta/m) \geqslant R(X) + \epsilon$, contradicting (1.5). $\square$

Remark 10. *In the above argument, to prove* $\delta(-R(X)K_X) \leqslant 1$*, one can also argue as follows. Suppose that* $\delta(-R(X)K_X) > 1$*. Then we may pick a sufficiently small rational number* $\epsilon > 0$ *such that* $R(X) + \epsilon \leqslant 1$ *and* $\delta\big(-(R(X)+\epsilon)K_X\big) \geqslant 1$*. Then it follows from [3, Corollary 2.11] that, the polarized pair* $\big(X, -(R(X)+\epsilon)K_X\big)$ *is K-semistable in the adjoint sense, hence twisted K-semistable in the sense of [13] (see [1, Proposition 8.2]). So [12, Proposition 10] guarantees that, for some* $\lambda \in (R(X), R(X)+\epsilon)$*, we can find two Kähler forms* $\omega, \alpha \in 2\pi c_1(X)$ *such that*

$$Ric(\omega) = \lambda\omega + (1-\lambda)\alpha,$$

which also gives us a contradiction.

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