Quantization proof of the uniform Yau-Tian-Donaldson conjecture

Kewei Zhang

Beijing Normal University

July 11-13, 2022

Outline

Notation

- X: a compact Kähler manifold of dimension n.
- ω: Kähler form on X with Kähler class $[\omega]$.
- ω^n: volume form on X, with total volume $V := \int_X \omega^n$.
- φ: ω-plurisubharmonic function on X.
- $dd^c := \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}$.
- $\mathrm{Ric}(\omega) := -dd^c \log\det\omega$, the Ricci form of ω.
- L: ample line bundle on X, with $mL = L^{\otimes m}$ for $m \in \mathbb{N}$.
- h: smooth Hermitian metric on L, with induced h^m on mL.
- $R_h := -dd^c \log h$, a smooth $(1,1)$-form representing $c_1(L)$.
- s: an element in $H^0(X, mL)$ for some $m \in \mathbb{N}$.
- $|s|^2_{h^m}$: the squared norm of s measured w.r.t. h^m.

1. Motivation

Monge–Ampère equations

General setup: let (X, ω) be an n-dimensional compact Kähler manifold. Let dV be a smooth positive volume form on X with

$$\int_X \omega^n = \int_X dV.$$

Put

$$\mathcal{H}_\omega := \left\{ \varphi \in C^\infty(X, \mathbb{R}) \middle| \omega_\varphi := \omega + dd^c\varphi > 0 \right\}.$$

Question

For $\lambda \in \mathbb{R}$, is there a Kähler potential $\varphi \in \mathcal{H}_\omega$ solving

$$(\omega + dd^c\varphi)^n = e^{-\lambda\varphi} dV?$$

Geometric meaning

By rescaling the Kähler class, may assume that $\lambda \in \{-1, 0, 1\}$. Suppose that $\varphi \in \mathcal{H}_\omega$ solves the above equation.

- $\lambda = -1$: ω_φ satisfies
$$\mathrm{Ric}(\omega_\varphi) = -\omega_\varphi + \theta$$
for some smooth form $\theta \in c_1(X) + [\omega]$.
- $\lambda = 0$: ω_φ satisfies
$$\mathrm{Ric}(\omega_\varphi) = \theta$$
for some smooth form $\theta \in c_1(X)$.
- $\lambda = 1$: ω_φ satisfies
$$\mathrm{Ric}(\omega_\varphi) = \omega_\varphi + \theta$$
for some smooth form $\theta \in c_1(X) - [\omega]$.

Note: all these metrics are twisted Kähler–Eisntein metrics.

Solvability

For the equation

$$(\omega + dd^c\varphi)^n = e^{-\lambda\varphi} dV$$

- $\lambda = -1$: can always be solved, by Yau and Aubin independently.
- $\lambda = 0$: can always be solved, by Yau's celebrated solution of the Calabi conjecture.
- $\lambda = 1$: There are obstructions related to K-stability/Ding stability.

In what follows we focus on the case of $\lambda = 1$. Our goal is understand what the Ding stability means and show the solvability of our equation under the stabilty assumption.

Several approaches

For some smooth form $\theta \in c_1(X) - [\omega]$, can apply

- Continuity method (Aubin, Yau, Tian,...)

$$\mathrm{Ric}(\omega_t) = t\omega_t + (1-t)\omega + \theta.$$

$$\Leftrightarrow (\omega + dd^c\varphi)^n = e^{-t\varphi_t} dV.$$

- Twisted Kähler–Ricci flow (Hamilton, Cao,...)

$$\frac{\partial}{\partial t}\omega_t = -\mathrm{Ric}(\omega_t) + \omega_t + \theta.$$

- Variational approach (Ding, Tian, Berman, Boucksom, Eyssidieux, Guedj, Zeriahi,...)
  Using Ding functional and pluri-potential theory.

We will adopt the last approach in this lecture series.

2. Analytic delta invariant

Ding functional

Define the Monge–Ampère energy by

$$E(\varphi) := \frac{1}{(n+1)\int_X \omega^n} \int_X \sum_{i=0}^{n} \varphi \omega^i \wedge \omega_\varphi^{n-i}, \ \varphi \in \mathcal{H}_\omega.$$

Consider the following functional:

$$D(\varphi) := -\log \int_X e^{-\varphi} dV - E(\varphi).$$

This is called Ding functional (first introduced by Prof. W.Y. Ding in the 80's). If φ is a critical point, then up to a constant it holds that

$$(\omega + dd^c\varphi)^n = e^{-\varphi} dV.$$

Properness/Coercivity

Definition

We say D is proper/coercive if there exist $\varepsilon > 0$ and $C > 0$ such that

$$D(\varphi) \geq \varepsilon(\sup \varphi - E(\varphi)) - C \text{ for any } \varphi \in \mathcal{H}_\omega.$$

Remark: one can replace $\sup \varphi - E(\varphi)$ with $J(\varphi)$ in the above definition.

Theorem (BBEGZ+Székelyhidi–Tosatti)

If D is proper, then D admits a minimizer $\varphi \in \mathcal{H}_\omega$ which solves

$$(\omega + dd^c\varphi)^n = e^{-\varphi} dV.$$

Optimal Moser–Trudinger exponent

Definition (Optimal exponent in Moser–Trudinger inequality)

The analytic δ-invariant of (X,ω) is defined to be

$$\delta^A([\omega]) := \sup\left\{\delta > 0 \middle| \sup_{\varphi\in\mathcal{H}_\omega}\int_X e^{-\delta(\varphi - E(\varphi))}dV < +\infty\right\}.$$

It does not depend on ω or dV. When $\omega \in c_1(L)$ for some ample $\mathbb{R}$-line bundle, we simply write

$$\delta^A(L) := \delta^A([\omega]).$$

Remark: the study of Moser–Trudinger inequalities on Kähler manifolds goes back to Aubin, Tian, Ding, Ding–Tian, Tian–Zhu, Phong–Song–Sturm–Weinkove, Berman, et. al.

Proposition

The Ding functional is proper iff $\delta^A([\omega]) > 1$.

Tian's alpha invariant

Another closely related invariant is the following one.

Definition (Tian, 1987)

The alpha invariant of (X, ω) is defined by

$$\alpha([\omega]) := \sup \left\{ \alpha > 0 \middle| \sup_{\varphi \in \mathcal{H}_\omega} \int_X e^{-\alpha(\varphi - \sup \varphi)} dV < +\infty \right\}.$$

Again, this invariant only depends on $[\omega]$ and it is always positive.

Theorem (Tian)

One has $\delta^A([\omega]) \geq \frac{n+1}{n} \alpha([\omega])$.

As a consequence

Theorem (Tian's alpha-criterion)

If $\alpha([\omega]) > \frac{n}{n+1}$, *then* D *admits a minimizer* $\varphi \in \mathcal{H}_\omega$ *which solves*

$$(\omega + dd^c \varphi)^n = e^{-\varphi} dV.$$

3. Algebraic delta invariant

Log canonical threshold

Given an effective divisor D on X, locally D is cut out by some holomorphic function f, namely,

$$D = \{f = 0\}.$$

The lct of D is then defined by

$$\operatorname{lct}(X, D) := \inf_{p \in X} c_p(f),$$

where

$$c_p(f) := \sup\left\{c > 0 \middle| \frac{1}{|f|^c} \in L^2_{loc} \text{ } around \text{ } p\right\},$$

which is called the complex singularity exponent of f at p.
Equivalently, choose any smooth Hermitian metric h on the line bundle $\mathcal{O}_X(D)$, one has

$$\operatorname{lct}(X, D) = \sup\left\{c > 0 \middle| \int_X \frac{1}{|s_D|_h^{2c}} dV < +\infty\right\},$$

where $s_D \in H^0(X, \mathcal{O}_X(D))$ is the defining section of D.

Log canonical threshold of $\mathbb{Q}$-divisors

An effective $\mathbb{Q}$-divisor D on X is a $\mathbb{Q}_{\geq 0}$-combination of prime divisors. Namely,

$$D = a_1 D_1 + \cdots + a_k D_k,$$

where $a_i \in \mathbb{Q}_{\geq 0}$ and each D_i is an irreducible hypersurface on X. We can pick $r \in \mathbb{N}$ to be sufficiently divisible so that $a_i r \in \mathbb{N}$ for all i. So rD is an effective divisor in the usual sense. Then define

$$\mathrm{lct}(X, D) := r\,\mathrm{lct}(X, rD).$$

Exercise: Show that this definition of $\mathrm{lct}(X, D)$ does not depend on the choice of r.
In what follows, unless otherwise specified, D will always be an effective $\mathbb{Q}$-divisor.

Divisorial valuations

Given a prime divisor $F \subset Y \xrightarrow{\pi} X$ over X, a valuation of the form $\lambda \mathrm{ord}_F$ for some $\lambda \in \mathbb{R}_+$ is called divisorial. Put

$$A_X(F) := 1 + \mathrm{ord}_F(K_Y - \pi^* K_X),$$

which is called the log discrepancy of F. Given any effective divisor D on X, one can make sense of $\mathrm{ord}_F(D)$ by putting

$$\mathrm{ord}_F(D) := \mathrm{coeff}_F(\pi^* D).$$

Then one has the following valuative characterization of log canonical threshold (by Hironaka's resolution of singularities):

$$\mathrm{lct}(X, D) = \inf_F \frac{A_X(F)}{\mathrm{ord}_F(D)}.$$

Algebraic delta invariant

Let (X, L) be a polarized manifold of dimension n. In 2016, Fujita–Odaka introduced the following algebraic invariant for (X, L). For $m \in \mathbb{Z}_{>0}$, consider a basis $s_1, \cdots, s_{d_m}$ of the vector space $H^0(X, mL)$, where $d_m := \dim H^0(X, mL)$. For this basis, consider the $\mathbb{Q}$-divisor

$$D := \frac{1}{md_m} \sum_{i=1}^{d_m} \{s_i = 0\} \sim_{\mathbb{Q}} L.$$

Any $\mathbb{Q}$-divisor D obtained in this way is called an m-basis type divisor of L. Let

$$\delta_m(L) := \inf \left\{ \operatorname{lct}(X, D) \middle| D \text{ is } m\text{-basis type of } L \right\}.$$

Then let

$$\delta(L) = \limsup_m \delta_m(L).$$

This limsup is in fact a limit by the work of Blum–Jonsson. Roughly speaking, $\delta(L)$ measures the **expectation** of the lct of elements in $|L|_{\mathbb{Q}}$.

Valuative formulation

Let $\pi : Y \to X$ be a proper birational morphism and let $F \subset Y$ be a prime divisor F in Y. Let

$$S_m(L,F) := \frac{1}{md_m} \sum_{j\geq 1} \dim H^0(Y, m\pi^*L - jF)$$

denote the *m-th expected vanishing order of L along F*. Then a basic but important linear algebra lemma due to Fujita–Odaka says that

$$S_m(L,F) = \sup\left\{\mathrm{ord}_F(D) \,:\, m\text{-basis divisor } D \text{ of } L\right\},$$

and this supremum is attained by any m-basis divisor D arising from a basis $\{s_i\}$ that is *compatible* with the filtration

$$H^0\big(m\pi^*L\big) \supset H^0\big(m\pi^*L - F\big) \supset \cdots \supset \{0\},$$

meaning that each $H^0(m\pi^*L - jF)$ is spanned by a subset of the $\{s_i\}_{i=1}^{d_m}$.

Valuative formulation

Fujita–Odaka show that

$$\delta_m(L) = \inf_F \frac{A_X(F)}{S_m(L,F)}.$$

As $m \to \infty$, one has

$$S(L,F) := \lim_{m\to\infty} S_m(L,F) = \frac{1}{\mathrm{vol}(L)} \int_0^{+\infty} \mathrm{vol}(\pi^* L - xF)dx,$$

which is called the *expected vanishing order of* L *along* F.
Then using Newton–Okounkov bodies, Blum–Jonsson further show that, the limit of $\delta_m(X)$ also exists, and is equal to

$$\delta(L) = \lim_{m\to\infty} \delta_m(L) = \inf_F \frac{A_X(F)}{S(L,F)}.$$

Special setting: Fano case

Assume that X is Fano, i.e., $-K_X$ is ample. In this case we simply denote

$$\delta(X) := \delta(-K_X).$$

Theorem (Fujita, Li, Blum–Jonsson, Berman–Boucksom–Jonsson, Liu–Xu–Zhuang,...)

The following are equivalent:

1. $\delta(X) > 1$;
2. X is (uniformly) K-stable;
3. X is (uniformly) Ding-stable.

Remark 1: K-stability was first introduced by Prof. Tian and later reformulated by Donaldson, which involves the famous Futaki invariant. Ding stability was introduced by Berman. These two stability notions are equivalent for Fanos.
Remark 2: For this reason, $\delta(X)$ is also called the stability threshold of X.
In 2016, Park–Won computed the δ-invariants of smooth del Pezzo surfaces using Newton polygons, whose estimates were improved by Cheltsov–Z in 2018.
More recently, Ahmadinezhad–Zhuang developed a general strategy to compute δ-invariants. The upshot is that, δ is very much computable!

General setting: polarized pair

For general polarized pair (X, L), may introduce

Definition (Blum–Jonsson)

We say (X, L) is uniformly Ding stable if $\delta(L) > 1$.

Under the Yau–Tian–Donaldson framework, can ask

A question of Yau–Tian–Donaldson type

Does uniform Ding stability imply the existence of twisted Kähler–Einstein metrics in $c_1(L)$, i.e., is the equation

$$(\omega + dd^c\varphi)^n = e^{-\varphi} dV$$

solvable when $\omega \in c_1(L)$?

The work of Berman–Boucksom–Jonsson

Theorem (BBJ (JAMS), 2018)

Assume that (X, L) is uniformly Ding stable and assume that there exists a semi-positive smooth representative $\theta \in c_1(X) - c_1(L)$, then one can find a Kähler form $\omega \in c_1(L)$ solving

$$\mathrm{Ric}(\omega) = \omega + \theta.$$

Remark 1: In the above setting, X is automatically Fano. Their proof replies crucially on the convexity of Mabuchi functional.
Remark 2: When $L = -K_X$ and $\theta = 0$, this result was first proved by Tian and Chen–Donaldson–Sun independently in 2012, as solution to the Yau–Tian–Donaldson conjecture.

Main result

One can actually remove the positivity assumption of BBJ.

Theorem (Z, 2021)

Assume that (X, L) is uniformly Ding stable. Then for any smooth representative $\theta \in c_1(X) - c_1(L)$, then one can find a Kähler form $\omega \in c_1(L)$ solving

$$\mathrm{Ric}(\omega) = \omega + \theta.$$

In other words, given a Kähler form $\omega \in c_1(L)$, the equation

$$(\omega + dd^c\varphi)^n = e^{-\varphi} dV$$

is solvable for any smooth volume form dV on X.

Remark 1: Our proof is completely different and does not need the convexity of Mabuchi functional or Ding functional. Moreover, our argument is simpler. The proof uses ideas going back to Tian's thesis.
Remark 2: Our result also holds when θ has mild singularities.

Key step

It suffices to prove the following

Theorem (Z, 2021)

One has $\delta(L) = \delta^A(L)$ for any ample line bundle L.

In other words, Fujita–Odaka's δ-invariant is precisely the optimal exponent of certain Moser–Trudinger type inequality.
As a consequence we obtain a uniform Yau–Tian–Donaldson theorem for the existence of twisted Kähler–Einstein metrics on general polarized manifolds.

A consequence: volume comparison

As a consequence of our main result, we obtain the following Kählerian analogue of Bishop's volume comparison theorem.

Theorem (Z, 2020, Volume Comparison)

Let (X, ω) be a Kähler manifold with

$$\mathrm{Ric}(\omega) \geq (n+1)\omega.$$

Then one has

$$\mathrm{vol}(X, \omega) \leq \mathrm{vol}(\mathbb{CP}^n, \omega_{FS}).$$

The equality holds if and only if (X, ω) is biholomorphically isometric to $(\mathbb{CP}^n, \omega_{FS})$. Here $\mathbb{CP}^n$ denotes the complex projective space and ω_{FS} is the Fubini–Study metric on $\mathbb{CP}^n$ such that $\int_{\mathbb{CP}^n} \omega_{FS}^n = 1$.

Remark: when $\omega \in c_1(X)$, this was first proved by K. Fujita.
Key points: Convexity of Ding functional and $\delta(L) \geq \delta^A(L)$.

Another consequence: cscK problem

We obtain a new and computable criterion for the existence of constant scalar curvature Kähler (cscK) metrics.

Theorem (Z, 2021)

Let L be an ample line bundle. Assume that $K_X + \delta(L)L$ is ample and $\delta(L) > n\mu(L) - (n-1)s(L)$, where $\mu(L) := \frac{-K_X \cdot L^{n-1}}{L^n}$ and $s(L) := \sup\{s \in \mathbb{R} | -K_X - sL \text{ ample}\}$. Then X admits a unique constant scalar curvature Kähler (cscK) metric in $c_1(L)$.

Remark: a Kähler metric is said to have constant scalar curvature if

$$R(\omega) := \operatorname{tr}_\omega \operatorname{Ric}(\omega)$$

is a constant. So in partuclar, a Kähler–Einstein metric has constant scalar curvature.

4. The proof: Quantization approach

Quantization

The quantization approach goes back to Tian's thesis work in 1989. Choose a smooth Hermitian metric h on L such that $\omega = -dd^c \log h \in c_1(X)$. Consider the vector space $H^0(X, mL)$ for $m \gg 1$. Put

$$\mathcal{B}_m := \left\{ \varphi = \frac{1}{m} \log \sum_{i=1}^{d_m} |s_i|^2_{h^m} \,\middle|\, \{s_i\} \text{ basis of } H^0(X, mL) \right\}.$$

Then $\mathcal{B}_m \subset \mathcal{H}_\omega$ is called the space of Bergman potentials. It is of finite dimension.

Theorem (Catlin–Lu–Tian–Zelditch et al.)

Any $\varphi \in \mathcal{H}_\omega$ can be approximated by a sequence of Bergman potentials $\varphi_m \in \mathcal{B}_m$ as $m \to \infty$.

Construction of φ_m

For any $\varphi \in \mathcal{H}_\omega$, consider the Hermitian inner product

$$H_m^\varphi(\cdot,\cdot) := \int_X (he^{-\varphi})^m(\cdot,\cdot)\omega^n$$

on the vector space $H^0(X, mL)$. Let $\{s_i\}_{i=1}^{d_m}$ be the H_m^φ-orthonormal basis of $H^0(X, mL)$. Then let

$$\varphi_m := \frac{1}{m}\log\sum_{i=1}^{d_m}|s_i|^2_{h^m}.$$

Note that φ_m is independent of the choice of $\{s_i\}$.
Remark: In the definition of H_m^φ one can replace ω^n by an arbitrary smooth volume form and it always holds that $\varphi_m \to \varphi$ in C^∞-topology as $m \to \infty$.

An important identity

A trivial but quite important fact about φ_m is given as follows:

$$\int_X e^{m(\varphi_m - \varphi)} \omega^n = d_m.$$

Indeed,

$$\begin{aligned}
\int_X e^{m(\varphi_m - \varphi)} \omega^n &= \int_X \Big(\sum_{i=1}^{d_m} |s_i|^2_{h^m}\Big) e^{-m\varphi} \omega^n \\
&= \sum_{i=1}^{d_m} \int_X (he^{-\varphi})^m (s_i, s_i) \omega^n \\
&= \sum_{i=1}^{d_m} H_m^{\varphi}(s_i, s_i) = d_m.
\end{aligned}$$

Quantized Monge–Ampère energy

Given

$$\varphi = \frac{1}{m} \log \sum_{i=1}^{d_m} |s_i|^2_{h^m} \in \mathcal{B}_m,$$

where $\{s_i\}$ is a basis of $H^0(X, mL)$, we put

$$E_m(\varphi) := \frac{1}{md_m} \log \det \left[\int_X h^m(s_i, s_j) \omega^n \right],$$

It was observed by Donaldson that, by using the Bergman asymptotics, one has

$$E_m(\varphi_m) \to E(\varphi) \text{ as } m \to \infty.$$

Quantized Moser–Trudinger inequality

Definition (Rubinstein–Tian-Z, 2020)

The analytic δ_m-invariant of (X, L) is defined by

$$\delta_m^A(L) := \sup\left\{\delta > 0 \,\middle|\, \sup_{\varphi \in \mathcal{B}_m} \int_X e^{-\delta(\varphi - E_m(\varphi))} dV < \infty\right\}.$$

This definition is very similar to Tian's α_m-invariants to be introduced below, the only difference being that $\sup\varphi$ is replaced by $E_m(\varphi)$.

Rubinstein–Tian–Z, 2020

One has

$$\delta_m^A(L) = \delta_m(L).$$

The α_m invariant

Definition (Tian, 1989)

The α_m-invariant of (X, L) is defined by

$$\alpha_m(L) := \sup \left\{ \alpha > 0 \middle| \sup_{\varphi \in \mathcal{B}_m} \int_X e^{-\alpha(\varphi - \sup \varphi)} dV < +\infty \right\}.$$

Proposition

One has $\alpha(L) = \lim_m \alpha_m(L)$.

This can be easily shown using the following consideration: for any $\varphi \in \mathcal{H}_\omega$ and $\alpha > 0$,

$$\int_X e^{-\alpha\varphi} dV \leq \left(\int_X e^{m(\varphi_m - \varphi)} dV \right)^{\alpha/m} \left(\int_X e^{-\frac{\alpha}{1-\alpha/m}\varphi_m} dV \right)^{1-\alpha/m}.$$

The strategy to prove $\delta^A = \delta$

Draw a diagram:

The proof of $\delta_m^A = \delta_m$

The proof of this has two steps. First, applying the lower semi-continuity of complex singularity exponent of Demailly–Kollár, we show that it suffices to consider basis divisors associated to **orthonormal** basis of $H^0(X, mL)$. Then geometric mean inequality

$$\sum_{i=1}^{d_m} \mu_i |s_i|^2_{h^m} \geq d_m \Big(\prod_{i=1}^{d_m} \mu_i \Big)^{\frac{1}{d_m}} \Big(\prod_{i=1}^{d_m} |s_i|^2_{h^m} \Big)^{\frac{1}{d_m}},$$

will give us

$$\delta_m^A(X, L) \geq \delta_m(X, L).$$

Second, a local computation around the center of a divisorial valuation gives us a uniform integral control along Bergman geodesics:

$$\int_X \frac{e^{tA_X(F)}}{\left(\sum_{i=1}^{d_m} e^{t\,\mathrm{ord}_F s_i} |s_i|^2_{h^m} \right)^{\frac{\delta}{m}}} dV > C > 0, \text{ for all } t \geq 0.$$

We combine this with the valuative description of $\delta_m(L)$ to conclude

$$\delta_m^A(X, L) \leq \delta_m(X, L).$$

Corollary: quantized Yau–Tian–Donaldson theorem

Rubinstein–Tian-Z, 2020

Assume that X is a Fano manifold, then the following assertions hold:

1. If $\delta_m(X) > 1$, then X admits an anti-canonical balanced metric;
2. If X admit an anti-canonical balanced metric, then $\delta_m(X) \geqslant 1$. If such a metric is unique, $\delta(X) > 1$.

This follows from a variational argument using quantized Ding functinoal and Berdtsson convexity.

The proof for $\delta^A = \lim_{m\to\infty} \delta_m^A$

It remains to show that

$$\delta^A(L) = \lim_{m\to\infty} \delta_m^A(L).$$

The proof splits into two steps:

Proposition (BBGZ)

For any $\varepsilon > 0$, there exists m_0 such that for any $m \geq m_0$,

$$E_m(\varphi) \leq (1-\varepsilon)E(\varphi) + \varepsilon \sup \varphi + \varepsilon \text{ for all } \varphi \in \mathcal{B}_m.$$

This gives $\delta^A(L) \leq \liminf_{m\to\infty} \delta_m^A(L)$.

Proposition (Z, 2021: weak partial C^0 estimate)

For any $\varepsilon > 0$, there exists m_0 such that for any $m \geq m_0$,

$$E(\varphi) \leq E_m(((1-\varepsilon)\varphi)_m) + \varepsilon \sup \varphi \text{ for all } \varphi \in \mathcal{H}_\omega.$$

This gives $\delta^A(L) \geq \limsup_{m\to\infty} \delta_m^A(L)$.
The proof is complete.

Summery: solvability of Monge–Ampère equations

Theorem (Z, 2021)

Let (X, L) be a polarized manifold and assume that it is uniformly Ding stable (i.e. $\delta(L) > 1$). Let $\omega \in c_1(L)$ be a Kähler form and dV be any smooth positive volume form. Then the Monge–Ampère equation

$$(\omega + dd^c\varphi)^n = e^{-\varphi} dV$$

always has a solution $\varphi \in \mathcal{H}_\omega$.

Remark: this result also holds for certain volume form dV with L^p density for some $p > 1$. For instance, conic KE problem:

$$(\omega + dd^c\varphi)^n = \frac{e^{-\varphi} dV}{||s||^{2(1-\beta)}}.$$

Thanks for your attention!