Convex bodies associated to Kähler manifolds

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Outline

- Background
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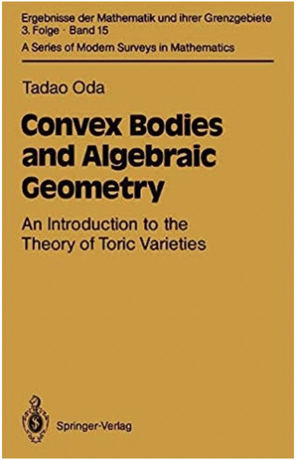
Background

Motivation



The relation between convex geometry and algebraic geometry has been explored intensively:

- Toric varieties and polytopes
- Intersection numbers and lattice points
- Big line bundles and Okounkov bodies

Big line bundles

Let X be a complex projective manifold of dimension n. Let L be a holomorphic line bundle on X. The volume of L is defined to be

$$\mathrm{vol}(L) := \limsup_{m\to\infty} \frac{\dim H^0(X, mL)}{m^n/n!}.$$

The line bundle L is said to be big if

$$\mathrm{vol}(L) > 0.$$

A line bundle being big means that mL has many global holomorphic sections. Big line bundles play crucial roles in birational geometry.

Okounkov bodies

In 1996, Okounkov gave an interesting construction as follows:
Let $Y_\bullet = \{Y_i\}_{i=0}^n$ be a flag of submanifolds:

$$X = Y_0 \supset Y_1 \supset ... \supset Y_n = \{p\},$$

where Y_i is a closed complex submanifold of codiemsnion i.
One can find a local holomorphic coordinate system $(z_1, ..., z_n)$ around p such that $Y_i = \{z_1 = ... = z_i = 0\}$.
Using the flag $Y_\bullet$ one can naturally define a valuative map from $H^0(X, mL)$ to the lattice $\mathbb{Z}^n$ in $\mathbb{R}^n$.

Okounkov bodies

For any non-zero holomorphic section $s \in H^0(X, mL)$, it is a locally a holomoprhic function, say $s = f(z_1, ..., z_n)$. One can expand it into Taylor series:

$$f(z_1, ..., z_n) = \sum_{\alpha \in \mathbb{N}^n} c_\alpha z^\alpha,$$

with $z^\alpha := \prod_{i=1}^n z_i^{\alpha_i}$. One define

$$\nu(s) := \min\{\alpha | c_\alpha \neq 0\} \in \mathbb{Z}^n,$$

where the min is taken with respect to the lexicographic order.

Okounkov body

More geometrically, the map $\nu : H^0(X, mL)\backslash\{0\} \to \mathbb{Z}^n$ can be described as follows.

One can define $\nu(s) = (\nu_1(s), ..., \nu_n(s))$ inductively. Put

$$\nu_1(s) := \mathrm{ord}_{Y_1}(s).$$

Then $s/z_1^{\nu_1(s)}$ defines a global section in $H^0(X, mL - \nu_1(s)Y_1)$, not identically zero along Y_1. Let s_1 be its restriction on Y_1, so it is a non-zero section in $H^0(Y_1, (mL - \nu_1(s)Y_1)|_{Y_1})$. Then put

$$\nu_2(s_1) := \mathrm{ord}_{Y_2}(s_1).$$

In this way one defines all the remain values $\nu_i(s)$.

Okounkov body

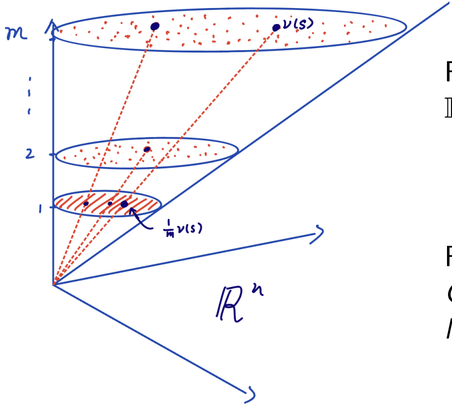
A simple linear algebra argument shows that

$$\#\{\nu(s)|s \in H^0(X, mL)\backslash\{0\}\} = \dim H^0(X, mL).$$

The Okounkov body $\Delta_{Y_\bullet}(L)$ is defined as

$$\Delta_{Y_\bullet}(L) := \overline{\bigcup_{m\geq 1} \frac{1}{m}\left\{\nu(s)\middle| s \in H^0(X, mL)\backslash\{0\}\right\}}.$$

The volume identity



Fact: $\Delta_{Y_\bullet}(L)$ is a convex body in $\mathbb{R}^n$ satisfying the volume identity:

$$\mathrm{vol}_{\mathbb{R}^n}(\Delta_{Y_\bullet}(L)) = \frac{1}{n!}\mathrm{vol}(L).$$

Reference: Lasarsfeld–Mustata, *Convex bodies associated to linear series*, ASENS, 2009.

Lasarsfeld–Mustata's question

Finally, asymptotic invariants of linear series have appeared in other settings, and it is natural to wonder whether the machinery developed here extends as well. Paoletti and others [37], [38], [13] have studied equivariant volume functions and related invariants in the presence of a group action. This suggests:

PROBLEM 7.6. – Extend the theory in the present paper to an equivariant setting.

The original paper [34] of Okounkov, as well as [35], [1], [28], [26], might be relevant. There has also been some very interesting recent work on arithmetic analogues of the volume function [43], [30], which leads to:

QUESTION 7.7. – Can one construct "arithmetic Okounkov bodies"? (17)

When X is a compact complex manifold, Boucksom [6], [7] has defined and studied the volume (and other invariants) of an arbitrary pseudo-effective $(1,1)$-class α on X. It is natural to wonder whether one can realize these volumes by convex bodies as well.

Main Result:
The same result holds in Kähler geometry!

Main result

Let (X, ω) be a compact Kähler manifold of dimension n. Here ω is a Kähler form: smooth reall d-closed positive $(1,1)$-form. Then its volume is

$$\mathrm{vol}(X, \omega) := \int_X \omega^n.$$

This quantity only depends on the Kähler class $\{\omega\} \in H^{(1,1)}(X, \mathbb{R})$.

Theorem

One can naturally associate a convex body $\Delta(\{\omega\})$ in $\mathbb{R}^n$ such that

$$\mathrm{vol}(X, \{\omega\}) = \frac{1}{n!} \mathrm{vol}_{\mathbb{R}^n}(\Delta(\{\omega\})).$$

Actually we prove a stronger result: the same holds for big classes.

Big cohomology class

Let (X, ω) be a compact Kähler manifold of dimension n. Let $\xi \in H^{1,1}(X, \mathbb{R})$ be a $(1,1)$ cohomology class. Let $\theta \in \xi$ be a smooth representative, i.e., θ is a smooth real d-closed $(1,1)$ form.

If θ is positive definite everywhere, then θ is a Kähler form and ξ is a Kähler class. But in what follows ξ is not assumed to be Kähler.

The class ξ is said to be big if ξ contains a Kähler current. Namely, there exists an L^1 integrable upper semi-continuous real valued function φ on X such that

$$\theta + dd^c\varphi \geq \varepsilon\omega$$

in the distribution sense for some $\varepsilon > 0$. Here $dd^c := \sqrt{-1}\partial\bar{\partial}/2\pi$. If L is a big line bundle, then $c_1(L)$ is a big class.

Kähler currents with analytic singularities

A famous regularization result due to Demailly says the following: given any Kähler current $T := \theta + dd^c\varphi$ in a big class ξ, one can find a decreasing sequence $\varphi_j \searrow \varphi$ such that

- e^{φ_j} is a non-negative Hölder continuous function on X.
- Each $T_j := \theta + dd^c\varphi_j$ satisfies $T_j \geq T - \varepsilon_j\omega$, with $\varepsilon_j \to 0$.
- φ_j has analytic singularities, namely, locally

$$\varphi_j = c\log\left(\sum_i |f_i|^2\right) + g$$

for $c \in \mathbb{Q}_{>0}$, $\{f_i\}$ a finite set of holomorphic functions and g is a bounded function. And φ_j is smooth away from $\{\varphi = -\infty\}$.

Therefore, there are many Kähler currents with analytic singularities in a big class.

Volume of big class

Let ξ be a big class. Then its volume is defined as

$$\mathrm{vol}(\xi) := \sup_{T \in \xi} \int_{X \setminus \mathrm{Sing}(T)} T^n,$$

where the sup is over all Kähler currents with anayltic singularities.

If ξ is a Kähler class, i.e., there is a Kähler form $\omega \in \xi$, then

$$\mathrm{vol}(\xi) = \int_X \omega^n.$$

If L is a big line bundle, then $\mathrm{vol}(L) = \mathrm{vol}(c_1(L))$.

Reference: Boucksom, *On the volume of a line bundle*, 2002.

Lelong numbers

Given a Kähler current $T = \theta + dd^c\varphi$, the function φ typically has many singularities, namely, $\{\varphi = -\infty\} \neq \emptyset$. The Lelong number measures how fast φ tends to $-\infty$. Let $p \in X$, the Lelong number of T (or φ) at p is defined to be

$$\nu(T, p) = \nu(\varphi, p) := \sup\{c \geq 0 | \varphi(x) \leq c \log \mathrm{dist}(x, p)^2 + O(1) \text{ near } p\}.$$

If Y is a submanifold of X, the Lelong number of T along Y is defined to be

$$\nu(T, Y) := \inf_{p \in Y} \nu(T, p).$$

Also define

$$E_+(T) := \{p \in X | \nu(T, p) > 0\}.$$

When $T = \theta + dd^c\varphi$ has analytic singularities, $E_+(T)$ is precisely the singular locus of φ: $E_+(T) = \{\varphi = -\infty\}$.

Siu's decomposition

If $Y \subset X$ is a submanifold of codimension 1, then Y itself defines a positive current $[Y]$ in the cohomology class $\{Y\} \in H^{1,1}(X, \mathbb{R})$:

$$([Y], \beta) := \int_Y \beta \text{ for any } (n-1, n-1) \text{ form } \beta.$$

Now, if $T \in \xi$ is a Kähler current, it may have positive Lelong number along Y. By Siu's decomposition,

$$T - \nu(T, Y)[Y] \text{ is also a Kähler current,}$$

but it has zero Lelong number along Y. If in addition T has analytic singularities, then $T - \nu(T, Y)[Y]$ is smooth almost everywhere near Y. So it makes sense to look at the restriction $(T - \nu(T, Y)[Y])|_Y$, which is a Kähler current on Y.

Convex bodies associated to big classes

Let $Y_\bullet = \{Y_i\}_{i=0}^n$ be a flag of submanifolds:

$$X = Y_0 \supset Y_1 \supset ... \supset Y_n = \{p\},$$

where Y_i is a closed complex submanifold of codiemsnion i.
For any Kähler current $T \in \xi$ with analytic singularities, one can define a valuative vector $\nu(T) = (\nu_1(T), ..., \nu_n(T)) \in \mathbb{R}^n$ as follows (due to Y. Deng). Put

$$\nu_1(T) := \nu(T, Y_1).$$

Then let $T_1 := (T - \nu_1(T)[Y_1])|_{Y_1}$ and put

$$\nu_2(T) := \nu(T_1, Y_2).$$

In this manner one defines all the remaining $\nu_i(T)$.

Convex bodies associated to big classes

Then the Okounkov body associted to the big class ξ is defined as

$$\Delta_{Y_\bullet}(\xi) := \overline{\{\nu(T) | T \in \xi \text{ Kähler current with analytic singularities}\}}.$$

Fact: $\Delta_{Y_\bullet}(\xi)$ is a convex body in $\mathbb{R}^n$.

Theorem (Y. Deng, 2017)

When $n = 2$, one has

$$\mathrm{vol}(\xi) = \frac{1}{2} \mathrm{vol}_{\mathbb{R}^2}(\Delta_{Y_\bullet}(\xi)).$$

Deng conjectured that the same holds for higher dimensions.

The volume identity

We prove Deng's conjecture. Namely, we verify the following.

Conjecture A_n

For any $n \geq 1$, any big class ξ and any flag $Y_\bullet$ on an n-dimensional compact Kähler manifold X, one has

$$\mathrm{vol}(\xi) = \frac{1}{n!}\,\mathrm{vol}_{\mathbb{R}^n}(\Delta_{Y_\bullet}(\xi)).$$

Remark: it is possible that X does not admits any flag $Y_\bullet$. But one can always blow up X to produce many flags.

The Proof

Easy case: dimension 1

When X is a compact Riemann surface, one has

$$H^{1,1}(X,\mathbb{R}) \cong H^2(X,\mathbb{R}) \cong \mathbb{R}.$$

Any class $\xi \in H^{1,1}(X,\mathbb{R})$ is completely determined by its degree

$$\deg_X \xi = ([X],\xi) = \int_X \xi.$$

The class is Kähler iff $\deg_X \xi > 0$ and $\mathrm{vol}(\xi) = \deg_X \xi$.
A flag $Y_\bullet$ on X is simply a point, say $p \in X$, and $\nu(T) = \nu(T,p)$ for a Kähler current T. In this case,

$$\Delta_{Y_\bullet}(\xi) = [0, \deg_X \xi].$$

So Conjecure A_n holds for $n = 1$.

Higher dimensions: induction

For higher dimensions, we argue by induction on the dimension. For this purpose, we need the following crucial volume formula due to Witt Nyström.

Theorem (Witt Nyström, 2021)

Let ξ be a big class. Let $Y \subset X$ be a submanifold of codimension 1. Assume that there exists a Kähler current $T \in \xi$ such that $\nu(T, Y) = 0$. Put $\tau_Y(\xi) := \sup\{x > 0 | \xi - x\{Y\} \text{ big}\}$. Then

$$\mathrm{vol}(\xi) = n \int_0^{\tau_Y(\xi)} \mathrm{vol}_{X|Y}(\xi - x\{Y\})dx.$$

Here $\mathrm{vol}_{X|Y}(\cdot)$ denotes the restricted volume of a class along Y.

Restricted volume

Let ξ be a big class.
Let $Y \subset X$ be a submanifold of dimension $k \geq 1$. Assume that there exists a Kähler current $T \in \xi$ such that $\nu(T, Y) = 0$.
The restricted volume of ξ along Y is defined as

$$\mathrm{vol}_{X|Y}(\xi) := \sup_{T \in \xi} \int_{Y \setminus \mathrm{Sing}(T)} (T|_Y)^k,$$

where the sup is over all Kähler currents with anayltic singularities.

Warning: It can happen that

$$\mathrm{vol}_{X|Y}(\xi) < \mathrm{vol}(\xi|_Y).$$

Reason: not every Kähler current on Y can be extended to a Kähler current on X.

Slice of Okounkov bodies

Let ξ be a big class and $Y_\bullet$ be a flag. Assume that there exists a Kähler current $T \in \xi$ such that $\nu(T, Y_1) = 0$. Put

$$\tau_{Y_1}(\xi) := \sup\{t > 0 | \xi - t\{Y_1\} \text{ big}\}.$$

Then $\Delta_{Y_\bullet}(\xi)$ is a convex body in $\mathbb{R}^n$ with first coordinate x_1 satisfies

$$x_1 \in [0, \tau_{Y_1}(\xi)].$$

Put the t-slice of $\Delta_{Y_\bullet}(\xi)$ to be

$$\Delta_{Y_\bullet}(\xi)_t := \Delta_{Y_\bullet}(\xi) \cap \{t\} \times \mathbb{R}^{n-1}.$$

It is clear that

$$\mathrm{vol}_{\mathbb{R}^n}(\Delta_{Y_\bullet}(\xi)) = \int_0^{\tau_{Y_1}(\xi)} \mathrm{vol}_{\mathbb{R}^{n-1}}(\Delta_{Y_\bullet}(\xi)_t) dt.$$

Reduce to study the volume of slices

Conjecture B_n

One has

$$\mathrm{vol}_{\mathbb{R}^{n-1}}(\Delta_{Y_\bullet}(\xi)_t) = \frac{1}{(n-1)!}\mathrm{vol}_{X|Y_1}(\xi - t\{Y_1\}).$$

Then one easily sees that

Theorem

Conjecture B_n implies Conjecture A_n.

Next we wish to show that A_{n-1} implies B_n.

Extension of Kähler currents

We show that, when ξ is Kähler, any Kähler current with analytic singularities on a submanifold Y can be extended to X.

Theorem

Let ξ be a Kähler class. Let $Y \subset X$ be a submanifold of dimension $k \geq 1$. Assume that there exists a Kähler current $T \in \xi|_Y$ with analytic singularities on Y, then it can be extended to a Kähler current $\tilde{T} \in \xi$ on X with analytic singularities.

This result strengthens the extension result of Collins–Tosatti.
One cannot expect such result to hold for big classes!

Partial Okounkov bodies

In order to reduce our proof to the Kähler case, we introduce the partial Okounkov bodies of ξ with respect to T.

Let ξ be a big class. For any Kähler current $T \in \xi$ with analytic singularities, we put

$$\Delta_{Y_\bullet}(\xi, T) := \overline{\{\nu(T') | T' \in \xi \text{ with } T' \text{ being more singular than } T\}}.$$

Write $T' = \theta + dd^c u$ and $T = \theta + dd^c v$. Then T being more singular than R means that $u \le v + O(1)$.

Conjecture C_n

One has

$$\int_{X \setminus \mathrm{Sing(T)}} T^n = \frac{1}{n!} \mathrm{vol}_{\mathbb{R}^n}(\Delta_{Y_\bullet}(\xi, T)).$$

Reduce to the Kähler case

Using resolution of singularities, we prove the following:

Theorem

Conjecure A_n and Conjecture C_n are equivalent.

The advantage of Conjecture C_n is that it allows us to reduce the prove to the Kähler case, so we can apply the extension result.

The most non-trivial missing piece in our induction proof is the following.

Theorem

Conjecture C_{n-1} implies Conjecture B_n.

End of proof

We have shown that

- A_1 holds.
- A_n and C_n are equivalent.
- C_{n-1} implies B_n.
- B_n implies A_n.

So by induction we see that A_n, B_n, C_n all hold true.

Thanks for your attention!