The Ricci iteration towards canonical metrics

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Outline

1. Motivation: Searching for canonical metrics

The Calabi problem

In the 50s, Calabi proposed to study canonical metrics on a compact Kähler manifold. Let (X, ω) be an n-dimensional compact Kähler manifold. Let $\{\omega\} \in H^{1,1}(X, \mathbb{R})$ denote the Kähler class of ω. Our goal is to find the "best" candidate $\omega^* \in \{\omega\}$, which is called the canonical metric in $\{\omega\}$.

Examples

- Kähler–Einstein (KE) metric
- Constant scalar curvature Kähler (cscK) metrics
- Extremal metrics

In what follows, we mainly focus on the first two cases.

Kähler–Einstein problem

Let (X, ω) be an n-dimensional compact Kähler manifold. Let $\{\omega\} \in H^{1,1}(X, \mathbb{R})$ denote the Kähler class of ω. Our goal is to find a Kähler metric $\omega_{KE} \in \xi$ such that

$$\mathrm{Ric}(\omega_{KE}) = \lambda \omega_{KE}$$

for some constant $\lambda \in \mathbb{R}$.
Such a metric is called a Kähler–Einstein (KE) metric.
To find the KE metric, it amounts to solving a PDE:

Question

For $\lambda \in \mathbb{R}$, is there a Kähler potential $\varphi \in \mathcal{H}_\omega$ solving

$$(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{-\lambda\varphi} dV?$$

Solvability

For the equation

$$\mathrm{Ric}(\omega_\varphi) = \lambda\omega_\varphi$$

- $\lambda < 0$: can always be solved, by Yau and Aubin independently.
- $\lambda = 0$: can always be solved, by Yau's celebrated solution of the Calabi conjecture.
- $\lambda > 0$: There are obstructions related to K-stability/Ding stability: Mastushima, Futaki, Tian, Ding–Tian, Tian–Zhu, Zhu–Wang, Tian, Chen–Donaldson–Sun, Chen–Sun–Wang, Berman–Boucksom–Jonsson, Li–Tian–Wang...

Several approaches

- Continuity method (Aubin, Tian et. al.)

$$\mathrm{Ric}(\omega_t) = t\omega_t + (1-t)\omega.$$

- Kähler–Ricci flow (Hamilton, Cao, Perelman, Tian–Zhu, Tian–Zhang et. al.)

$$\partial_t \omega_t = -\mathrm{Ric}(\omega_t) + \omega_t.$$

- Variational approach (Ding, Tian, Berman, Boucksom, Eyssidieux, Guedj, and Zeriahi)
  Using Ding functional and pluri-potential theory.
- Ricci iteration (Rubinstein et. al.)

Ricci iteration: first glance

Theorem (Calabi–Yau Theorem)

For any smooth representative $\rho \in 2\pi c_1(X)$, *there exists a unique* $\omega_\varphi \in \{\omega\}$ *such that*

$$\mathrm{Ric}(\omega_\varphi) = \rho.$$

Assume that $\{\omega\} = 2\pi c_1(X)$, then given $\omega_0 \in \{\omega\}$, we can find

$$\mathrm{Ric}(\omega_1) = \omega_0,\ \mathrm{Ric}(\omega_2) = \omega_1,\ \ldots,\ \mathrm{Ric}(\omega_{i+1}) = \omega_i, \ldots$$

This sequence $\{\omega_i\}_{i\in\mathbb{N}}$ is called Ricci iteration. If the sequence smoothly converge to a limit ω_∞, then $\mathrm{Ric}(\omega_\infty) = \omega_\infty$. So we get a KE metric.

Question

Under what conditions do we have the convergence of ω_i?

Ricci iteration: more general version

Assume that $2\pi c_1(X) = \lambda\{\omega\}$, $\lambda \in \mathbb{R}$. Consider the normalized Kähler Ricci flow:

$$\partial_t \omega_t = -\operatorname{Ric}(\omega_t) + \lambda \omega_t.$$

This flow was first studied by Cao.

Theorem (Cao 1985)

When $\lambda \leq 0$, the flow $\{\omega_t\}_{t \geq 0}$ smoothly converges to a limit ω_∞ that satisfies $\operatorname{Ric}(\omega_\infty) = \lambda \omega_\infty$.

One can discretize the flow: for $\tau > 0$, consider

$$\frac{\omega_{i+1} - \omega_i}{\tau} = -\operatorname{Ric}(\omega_{i+1}) + \lambda \omega_{i+1}.$$

This is the more general Ricci iteration introduced by Rubinstein in 2007.

Ricci iteration: known results

This iteration sequence is equivalent to

$$\mathrm{Ric}(\omega_{i+1}) = (\lambda - \frac{1}{\tau})\omega_{i+1} + \frac{1}{\tau}\omega_i.$$

This is a twisted Kähler–Einstein equation.

Theorem (Rubinstein 2007)

When $\lambda \leq 0$, the iteration $\{\omega_i\}_{i\in\mathbb{N}}$ exists for any $\tau > 0$ and smoothly converges to a limit ω_∞ that satisfies $\mathrm{Ric}(\omega_\infty) = \lambda\omega_\infty$.

The case $\lambda > 0$ is more subtle. The sequence $\{\omega_i\}$ exists for small enough $\tau > 0$, but not all $\tau > 0$.

Theorem (Darvas–Rubinstein 2019)

Assume that $2\pi c_1(X) = \{\omega\}$ and there exists a KE metric $\omega^* \in \{\omega\}$. Then the iteration $\{\omega_i\}$ exists for all $\tau > 0$ and there exists $g_i \in \mathrm{Aut}(X, J)$ such that $g_i^*\omega_i$ smoothly converges to ω^*.

The cscK problem

Let (X, ω) be an n-dimensional compact Kähler manifold. Our goal is to find a Kähler metric $\omega^* \in \{\omega\}$ such that

$$R(\omega^*) = \mathrm{tr}_{\omega^*} \mathrm{Ric}(\omega^*) = \bar{R},$$

where $\bar{R} = 2\pi n \frac{c_1(X)\cdot\{\omega\}^{n-1}}{\{\omega\}^n}$ is the average of the scalar curvature. Such a metric is called a constant scalar curvature Kähler (cscK) metric.

To find the cscK metric, it amounts to solving a coupled system of equations:

$$\begin{cases} (\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^F \omega^n, \\ \Delta_{\omega_\varphi} F = \mathrm{tr}_{\omega_\varphi} \mathrm{Ric}(\omega) - \bar{R}. \end{cases}$$

The cscK metric: another viewpoint

We recall a well known fact:

Lemma

A closed $(1,1)$-form θ on X is harmonic with respect to ω if and only if $\mathrm{tr}_\omega\theta$ is constant.

Therefore, ω^* is cscK if and only if $\mathrm{Ric}(\omega^*)$ is a harmonic form with respect to ω^*.
For any Kähler form ω, we let $\mathrm{HRic}(\omega)$ denote the harmonic part of $\mathrm{Ric}(\omega)$. The goal is make the difference $\mathrm{Ric}(\omega) - \mathrm{HRic}(\omega)$ as small as possible.

A modified Kähler Ricci flow

We consider the following flow:

$$\partial_t \omega_t = -\operatorname{Ric}(\omega_t) + \operatorname{HRic}(\omega_t).$$

The limit, if exists, is a cscK metric. This flow has been studied by many authors: Guan, Simanca, Rubinstein, Chen–Zheng et. al. If $2\pi c_1(X) = \lambda\{\omega\}$, then $\operatorname{HRic}(\omega_t) = \lambda\omega_t$. And the flow reduces to the normalized Kähler Ricci flow of Cao.

Theorem (Chen–Zheng, 2013)

This flow has short time existence.

But the long time existence and limiting behavior of this flow are still open.

Distretization of the flow

In 2007, Rubinstein proposed to disretize the flow: for $\tau > 0$, consider

$$\frac{\omega_{i+1} - \omega_i}{\tau} = -\operatorname{Ric}(\omega_{i+1}) + \operatorname{HRic}(\omega_{i+1}).$$

This is a Ricci iteration further generalizing the previous ones in the KE case.

Conjecture (Rubinstein 2007)

Assume that there exists a cscK metric $\omega^* \in \{\omega\}$, then the sequence $\{\omega_i\}$ exists and converges to ω^* in a suitable sense.

Main results

We verify this conjecture. More precisely, we prove

Theorem A (Z, 2023)

There exists a uniform constant $\tau_0 \in (0, \infty]$, depending only on X and the Kähler class $\{\omega\}$, such that for any $\tau \in (0, \tau_0)$ the iteration sequence $\{\omega_i\}_{i\in\mathbb{N}}$ exists for all $i \in \mathbb{N}$, with each ω_i being uniquely determined by ω_0, along which Mabuchi's K-energy decreases.

Theorem B (Z, 2023)

Let (X, ω) be a compact Kähler manifold admitting a cscK metric in $\{\omega\}$. Then for any $\tau > 0$ the iteration sequence $\{\omega_i\}_{i\in\mathbb{N}}$ sequence exists and there exist holomorphic diffeomorphsims g_i such that $g_i^*\omega_i$ converges smoothly to a cscK metric.

Twisted cscK equation

Observe that the above Ricci iteration is equivalent to

$$R(\omega_{i+1}) = \bar{R} - \frac{n}{\tau} + \frac{1}{\tau}\mathrm{tr}_{\omega_{i+1}}\omega_i,$$

which is a twisted cscK equation. It is equivalent to

$$\begin{cases} (\omega + \sqrt{-1}\partial\bar{\partial}\varphi_{i+1})^n = e^{F_{i+1}}\omega^n, \\ \Delta_{i+1}(F_{i+1} + u_i) = \mathrm{tr}_{i+1}(\mathrm{Ric}(\omega) - \frac{1}{\tau}\omega) + \frac{n}{\tau} - \bar{R}. \end{cases} \tag{1}$$

Some energy functionals

Let (X,ω) be a compact Kähler manifold of dimension n, and set

$$\mathcal{H}_\omega := \{\varphi \in C^\infty(X,\mathbb{R}) | \omega_\varphi := \omega + dd^c\varphi > 0\}.$$

Put $V := \int_X \omega^n$. For any $u, v \in \mathcal{H}_\omega$, define

$$I(u,v) = I(\omega_u,\omega_v) := \frac{1}{V}\int_X (v-u)(\omega_u^n - \omega_v^n).$$

$$E(u,v) := \frac{1}{(n+1)V}\int_X (v-u)\sum_{i=0}^{n}\omega_u^i \wedge \omega_v^{n-i}.$$

$$J(u,v) = J(\omega_u,\omega_v) := \frac{1}{V}\int_X (v-u)\omega_u^n - E(u,v).$$

$$Ent(u,v) = Ent(\omega_u,\omega_v) := \frac{1}{V}\int_X \log\frac{\omega_v^n}{\omega_u^n}\omega_v^n.$$

Note that by Jensen's inequality, one has $Ent(u,v) \geq 0$.

Some energy functionals

For any closed $(1,1)$ form χ, define

$$\mathcal{J}^{\chi}(u,v) := \frac{1}{V}\int_X (v-u)\chi \wedge \sum_{i=0}^{n-1} \omega_u^i \wedge \omega_v^{n-1-i} - \bar{\chi} E(u,v),$$

where

$$\bar{\chi} := \frac{n}{V}\int_X \chi \wedge \omega^{n-1} = n\frac{\{\chi\}\cdot\{\omega\}^{n-1}}{\{\omega\}^n}.$$

The K-energy is defined by

$$K(u,v) = K(\omega_u, \omega_v) := Ent(u,v) + \mathcal{J}^{-\operatorname{Ric}(\omega_u)}(u,v).$$

The χ-twisted K-energy is

$$K^{\chi}(u,v) = K^{\chi}(\omega_u,\omega_v) := K(u,v) + \mathcal{J}^{\chi}(u,v).$$

Variational formula

One has the following variation formulas (for any $u, v \in \mathcal{H}_\omega$ and $f \in C^\infty(X, \mathbb{R})$):

$$\begin{cases} \frac{d}{dt}\Big|_{t=0} E(u, v+tf) = \frac{1}{V}\int_X f\omega_v^n, \\ \frac{d}{dt}\Big|_{t=0} \mathcal{J}^\chi(u, v+tf) = \frac{1}{V}\int_X f(\mathrm{tr}_{\omega_v}\chi - \bar{\chi})\omega_v^n, \\ \frac{d}{dt}\Big|_{t=0} K^\chi(u, v+tf) = \frac{1}{V}\int_X f(\bar{R} - \bar{\chi} + \mathrm{tr}_{\omega_v}\chi - R(\omega_v))\omega_v^n. \end{cases}$$

Therefore, if v is a critical point of $K^\chi(u, \cdot)$, then it satisfies the twisted cscK equation:

$$R(\omega_v) = \bar{R} - \bar{\chi} + \mathrm{tr}_{\omega_v}\chi.$$

Properness

Definition

Given $u \in \mathcal{H}_\omega$, we say $K^\chi(u, \cdot)$ is proper if there exists $\varepsilon > 0$ and $C > 0$ such that

$$K^\chi(u, v) \geq \varepsilon(I - J)(u, v) - C \text{ for any } v \in \mathcal{H}_\omega.$$

Remark: one should view $I - J$ as a distance function on $\mathcal{H}_\omega$.

Theorem (Chen–Cheng 2020)

Assume that $\chi \geq 0$. If $K^\chi(u, \cdot)$ is proper, then $K^\chi(u, \cdot)$ admits a minimizer $\varphi \in \mathcal{H}_\omega$ which solves

$$R(\omega_v) = \bar{R} - \bar{\chi} + \mathrm{tr}_{\omega_v}\chi.$$

Existence of Ricci iteration

For the Ricci iteration equation

$$R(\omega_{i+1}) = \bar{R} - \frac{n}{\tau} + \frac{1}{\tau}\mathrm{tr}_{\omega_{i+1}\omega_i},$$

we need to find a critical point for the twisted K-energy $K^{\omega_i/\tau}$. Namely we choose $\chi = \omega_i/\tau$.

Key Fact

One has

$$\mathcal{J}^{\omega_i/\tau}(\omega_i, \cdot) = \frac{1}{\tau}(I - J)(\omega_i, \cdot).$$

Therefore $K^{\omega_i/\tau}$ is proper for small enough $\tau \geq 0$. This proves the existence of $\{\omega_i\}$.

Decreasing of K-energy

Since ω_{i+1} minimizes the twisted K-energy $K^{\omega_i/\tau}$, one has

$$K(\omega_i, \omega_{i+1}) + \frac{1}{\tau}(I - J)(\omega_i, \omega_{i+1}) = K^{\frac{\omega_i}{\tau}}(\omega_i, \omega_{i+1}) \leq K^{\frac{\omega_i}{\tau}}(\omega_i, \omega_i) = 0.$$

This implies that

$$K(\omega, \omega_{i+1}) \leq K(\omega, \omega_i).$$

So the proof of Theorem A is complete.
The proof of Theorem B is a bit more complicated. If there exists a cscK metric $\omega^* \in \{\omega\}$, then the K-energy is bounded from below. Then the iteration $\{\omega_i\}$ exists for all $\tau > 0$. The convergence part requires a priori estimates for the equation (1).

A priori estimates

Recall the equation

$$\begin{cases} (\omega + \sqrt{-1}\partial\bar{\partial}\varphi_{i+1})^n = e^{F_{i+1}}\omega^n, \\ \Delta_{i+1}(F_{i+1} + u_i) = \mathrm{tr}_{i+1}(\mathrm{Ric}(\omega) - \frac{1}{\tau}\omega) + \frac{n}{\tau} - \bar{R}. \end{cases}$$

Following Chen–Cheng, assume the existence of cscK, one can derive

1. C^0 estimate,
2. $W^{2,p}$ estimate,
3. C^2 estimate,
4. higher regularities.

These estimates allow us to conclude Theorem B.

Thanks for your attention!