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题目：代数曲面上δ-不变量的精确计算

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摘要

本文将重点研究Fano流形上的δ-不变量与Kähler-Einstein度量的存在性之间的关系。δ-不变量与田刚教授引入的α-不变量和K-稳定性有着密切的联系。在论文的第一部分，我们讨论δ-不变量与α-不变量的性质，并证明δ-不变量与流形上的最大Ricci下界之间的等价关系，这推广了Fujita-Odaka与Blum-Jonsson的相关结果。在论文第二部分，我们研究如何在复曲面上有效计算δ-不变量。我们的方法是利用复曲面上的相交数不等式来控制曲面上除子的奇点。特别的，我们将计算三次曲面的δ-不变量并重新证明该类曲面的K-稳定性。同时我们也将计算一类log Fano曲面δ-不变量，从而得到一系列新的log K-稳定的复曲面的例子。

关键词：δ-不变量；Kähler-Einstein度量；K-稳定性；α-不变量

ABSTRACT

Computing delta invariants on algebraic surfaces

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This thesis mainly studies the relation between the δ-invariant and the existence of Kähler-Einstein metrics on Fano manifolds. δ-invariant is closely related to the α-invariant and the K-stability introduced by Tian. In the first part of the thesis, we discuss the properties of α and δ-invaraints. We will show the equivalence between the δ-invariant and the greatest Ricci lower bound on Fano manifolds, which generalizes a result of Fujita-Odaka and Blum-Jonsson. In the second part of this thesis, we will search for effective methods to calculate δ-invariant on complex surfaces. The main ingredients of our method is to estimate the singularities of divisors on surfaces via local intersection inequalities. In particular, we will calculate the δ-invariants and give a new proof of the K-stability of cubic surfaces. Meanwhile, we will also give a new family of log K-stable surfaces by calculating their δ-invariants.

KEYWORDS: δ-invariant; Kähler-Einstein metrics; K-stability; α-invariant

Content

Chapter 1 Introduction **1**

1.1 Kähler-Einstein problem 1

1.2 α-invaraint and δ-invariant 2

1.3 Our main results 4

Chapter 2 Preliminaries **7**

2.1 Canonical singularities in birational geometry 7

2.2 A tool box for complex surfaces 8

2.3 Some new local inequalities on complex surfaces 12

2.4 Tian's alpha invariant 19

Chapter 3 Stability thresholds on Fano type manifolds **23**

3.1 Fujita-Odaka's invaraint 23

3.2 Delta invaraint and the greatest Ricci lower bound 24

3.3 Analytic delta invariant 30

Chapter 4 Delta invariants on smooth cubic surfaces **33**

4.1 Multiplicity estimates 33

4.2 Proof of the main result 47

4.2.1 Case 1 48

4.2.2 Case 2 50

4.2.3 Case 3 52

4.2.4 Case 4 54

Chapter 5 Delta invariants on asymptotically log Fano surfaces **59**

5.1 Basic set-up 59

5.2 Multiplicity estimates 60

5.3 Proof of the main result 64

References **71**

致谢 **77**

北京大学学位论文原创性声明和使用授权说明 79

Chapter 1 Introduction

1.1 Kähler-Einstein problem

A central problem in Kähler geometry is to find canonical metrics on a given compact Kahler manifold. One important class of canonical metrics is the Kähler-Einstein (KE) metric. A Kähler metric is KE if the Ricci form of the Kähler metric is a constant multiple of the Kähler form. As we know, the Ricci form of a Kähler metric must lie in the first Chern class of the manifold. Therefore, a necessary condition for the existence of KE metric is that the first Chern class of the manifold has a sign.

The study of KE metrics has a long history. In the cases where the first Chern class is zero or negative, the uniqueness of the KE metric was proved by Calabi in the 1950s, and the existence of such a metric was obtained in 1978 by Yau [61] (see also Aubin [1]). However, when the first Chern class is positive (i.e., for Fano manifolds), the situation is much more complicated. It turns out that there are obstructions to the existence of KE metrics on Fano manifolds. The first obstruction was found by Matsushima [41] in 1957, which says that the automorphism group of a KE Fano manifold must be reductive. In 1983 another obstruction was found by Futaki in [31], where he defined an holomorphic invariant (which we now call Futaki invariant) and it was shown that the Futaki invariant must vanish if the Fano manifold admits a KE metric. In 1985, Bando-Mabuchi [9] showed that, if any, the KE metric on a Fano manifold is unique up to biholomorphic automorphisms.

So it is natural to ask, when does a Fano manifold admit a KE metric? Regarding this problem, many significant results were obtained in history. For instance, in 1990, Tian [53] completely solved the existence problem for Fano surfaces and showed that the existence of KE metrics is equivalent to the reductivity of the automorphism groups of Fano surfaces; in 2004, Xujia Wang and Xiaohua Zhu [60] showed that there always exist Kähler-Ricci solitons on toric Fano manifolds and the soliton metric is KE if and only if the Futaki invaraint vanishes.

For general Fano manifolds, the existence of KE metrics is more difficult to characterize. In 1992, Ding-Tian [22] defined a generalized Futaki invariant for a deformation family of Fano manifolds, and based on this, in 1997, Tian [55] introduced an algebro-geometric notion called K-stability. This notion was later reformulated by Donaldson [23] using more algebraic language. And the famous Yau–Tian–Donaldson conjecture says that, the existence of KE metrics on Fano manifolds is equivalent to K-stability. This conjecture was recently solved by Tian [58] and Chen-Donaldson-Sun [17] independently in 2012.(For the precise definition of K-stability, we refer the reader to [5, 23, 55].)

1.2 α-invaraint and δ-invariant

However, given a general Fano manifold, it is very difficult to test its K-stability and hence the existence of KE metric is not easy to determine. So it is an important problem to find a computable criterion that one can use to determine if the manifold admits a KE metric or not. In history, the first effective criterion was found in 1987 by Tian [52], which is known as the α-invariant.

Theorem 1.1 ([52]). *Let X be a Fano manifold. Suppose that $\alpha(X) > \frac{\dim(X)}{\dim(X)+1}$. Then X admits a Kähler-Einstein metric.*

Note that many examples of KE manifolds have been found with the help of α-invariant. Here, $\alpha(X)$ can be defined by

$$\alpha(X) = \sup\left\{\lambda \in \mathbb{Q}\ \middle|\ \begin{array}{l}\text{the log pair } (X, \lambda D) \text{ is log canonical}\\ \text{for every effective } \mathbb{Q}\text{-divisor } D \sim_{\mathbb{Q}} -K_X\end{array}\right\}.$$

Roughly speaking, $\alpha(X)$ measures the singularities of all the divisors in the pluri-anticanonical system. (See Section 2.4 for more details.) But as one can see, Theorem 1.1 only gives a sufficient condition for the existence of KE metrics and the condition $\alpha(X) > \frac{\dim(X)}{\dim(X)+1}$ turns out to be rather restrictive.

For instance, when X is a smooth cubic surface (which is a two-dimensional Fano manifold), it is possible that $\alpha(X) = \frac{2}{3}$ (cf. Example 2.23), so Theorem 1.1 fails to work in this case. However Tian [53] still managed to show the existence of KE metrics on cubic surfaces by modifying α-invariants and using hard core analysis. In Tian's argument, the singularities of pluri-anticaonical divisors play an essential role. So in his 1990 survey [54] (page 590), Tian wrote down the following expectation:

The author believes that the existence of Kähler-Einstein metric with positive scalar curvature should be closely related to the geometry of pluri-anticaonical divisors.

Recently, this expectation has been realized with the help of a new invariant introduced by Fujita-Odaka [26] in 2016, which we now describe. For a sufficiently large integer k, consider a basis $s_1, \cdots, s_{d_k}$ of the vector space $H^0(\mathcal{O}_X(-kK_X))$, where $d_k = h^0(\mathcal{O}_X(-kK_X))$. For this basis, consider $\mathbb{Q}$-divisor

$$\frac{1}{kd_k}\sum_{i=1}^{d_k}\left\{s_i = 0\right\} \sim_{\mathbb{Q}} -K_X.$$

Any $\mathbb{Q}$-divisor obtained in this way is called a k-basis type (anticanonical) divisor. Let

$$\delta_k\big(X\big) = \sup\left\{\lambda \in \mathbb{Q}\ \middle|\ \begin{array}{l}\text{the log pair } (X, \lambda D) \text{ is log canonical} \\ \text{for every } k\text{-basis type } \mathbb{Q}\text{-divisor } D \sim_{\mathbb{Q}} -K_X\end{array}\right\}.$$

Then let

$$\delta(X) = \limsup_{k\in\mathbb{N}} \delta_k(X).$$

So roughly speaking, $\delta(X)$ measures the singularities of basis type anti-canonical divisors. Then Tian's expectation can now be stated more rigorously using the following recent result obtained by Blum-Jonsson [4] in 2017.

Theorem 1.2 ([4, Theorem B]). *The following assertions hold:*

1. *X is K-semistable if and only if $\delta(X) \geqslant 1$;*
2. *X is uniformly K-stable if and only if $\delta(X) > 1$.*

Namely, δ-invariant serves as a criterion for the existence of KE metrics.

This result also has a natural extension to the log Fano setting. To be more precise, let (X, Δ) be a log Fano pair (i.e. (X, Δ) is a klt pair and $-K_X - \Delta$ is ample), then we can define log K-stability for this pair. And by the solution of YTD conjecture [17, 58] (see also the recent work of Tian-Wang [59]), we know that the geometric interpretation of the log K-stability is the existence of Kähler-Einstein edge metrics. (The Kähler-Einstein edge metric is a smooth KE metric on $X\backslash\Delta$ but with edge singularities along the divisor Δ.) Meanwhile, one can also define a log δ-invariant $\delta(X, \Delta)$ for the log pair (X, Δ) (cf. Definition 3.8). Then relying on the work of Blum-Jonsson [4], Codogni-Patakfalvi [18] showed the following in 2018.

Theorem 1.3. *([18, Corollary 4.8]) One has*

1. *(X, Δ) is log K-semistable if and only if $\delta(X, \Delta) \geqslant 1$;*

2. (X, Δ) is uniformly log K-stable if and only if $\delta(X, \Delta) > 1$.

In other words, the log δ-invariant can be used as a criterion for the existence of Kähler-Einstein edge metrics.

1.3 Our main results

In this thesis we will discuss several aspects of the δ-invariant. Our first purpose is to generalize Theorem 1.2 to the case of $\delta(X) < 1$. More precisely, we prove the following.

Theorem 1.4 ([12]). *Let X be a Fano manifold. let $\beta(X)$ denote the greatest Ricci lower bound of X. Then we have $\beta(X) = \min\{\delta(X), 1\}$.*

To prove this result, we will approximate $\beta(X)$ by a sequence of Kähler-Einstein edge metrics and it turns out that the corresponding log δ-invaraints of this sequence will converge to $\delta(X)$. Note that Theorem 1.4 can be thought of a special version of the Yau-Tian-Donaldson correspondence for twisted KE metrics.

We will also give an effective method to estimate δ-invariant on complex surface. Note that the computation of δ-invaraint is much harder than that of α-invariant. For α-invariant, one can use various tools from birational geometry to estimate the log canonical thresholds of divisors. For instance, the α-invariants of Fano surfaces have been explicitly computed by Cheltsov [11], and for Fano threefolds, this is also well studied in the work of Cheltsov-Shramov [13].

However, to calculate δ-invariant, so far there are very few tools. But still, there have been some important progress in this area. In [43], Park and Won estimated the δ-invariants of all smooth Fano surfaces using deep analysis of Newton-polygons. Their work gives a purely algebraic proof of Tian's work [53], but it seems that their method cannot be easily generalized to higher dimensions. For toric varieties, Blum-Jonsson [4] showed that the δ-invaraint can be completely determined by the barycenter of the corresponding polytope. However, their method is not likely to work for non-toric varieties.

So in this thesis we present an alternative and more geometric approach to estimating the δ-invariant. We will mainly work on complex surfaces, since this is already quite difficult. The main ingredient of our approach is the volume estimates for basis type divisors on surfaces (cf. Theorem 2.12), which can control the multiplicities of divisors locally and hence allows us to estimate the corresponding log canonical thresholds. In particular, we prove

Theorem 1.5 ([16]). *Let S be a smooth cubic surface in $\mathbb{P}^3$. Then $\delta(S) \geqslant \frac{6}{5}$.*

Note that this is a joint work with Ivan Cheltsov and it gives a new algebraic proof for the K-stability of smooth cubic surfaces. Moreover, our bound for $\delta(S)$ is better than the one obtained by Park-Won [43] We hope that out methods can be generalized to higher dimensions in future research.

Motivated by a conjecture of Cheltsov-Rubinstein [14], we will also investigate a special family of log Fano surfaces $(S,(1-\beta)C)$ (see section 5.1 for detailed definition of the family). In [14], this family is conjectured to admit Kähler-Einstein edge metrics, but it seems to the author that there is no easy analytic proof for this. So we attack this problem from the algebraic side. Namely we will try to show that this family is log K-stable by estimating the log δ-invariants. But it turns out that the boundary term $(1-\beta)C$ causes new troubles. To overcome this, we will prove several new local intersection inequalities in Section 2.3, which allows us to estimate the log canonical thresholds effectively even with the appearance of the boundary term $(1-\beta)C$, and thanks to which, we are able to prove the following.

Theorem 1.6 ([12]). *One has $\delta(S,(1-\beta)C) > 1$ for sufficiently small β.*

This is a joint work with Ivan Cheltsov and Yanir Rubinstein and it gives a whole new family of log K-stable surfaces. This partially verifies the conjecture of Cheltsov-Rubinstein [14].

The rest of this thesis is organized as follows. In Chapter 2, we collect some algebraic tools that will be useful for us. Several new local intersection inequalities are proved in Section 2.3, which will play significant roles in our computation of δ-invariants on complex surfaces. In Chapter 3 we will discuss the properties of α- and δ-invaraints with more detail and Theorem 1.4 will be proved in Section 3.2. In Chapter 4, we prove Theorem 1.5. In Chapter 5 we prove Theorem 1.6.

Chapter 2 Preliminaries

This chapter serves as a quick tour guide for various algebraic notions that will be used in this thesis. Moreover we will develop some new intersection formulae, which will play crucial roles in Chapter 4 and 5.

2.1 Canonical singularities in birational geometry

In this section we recall some basic terminologies for canonical singularities appearing in birational geometry. All varieties here are assumed to be normal over $\mathbb{C}$.

Given a proper birational morphism $\pi : Y \to X$, we define the exceptional set of π to be the smallest subset $\mathrm{exc}(\pi) \subset Y$, such that $\pi : Y \backslash \mathrm{exc}(\pi) \to X \backslash \pi(\mathrm{exc}(\pi))$ is an isomorphism.

A log resolution of (X, Δ) is a proper birational morphism $\pi : Y \to X$ such that $\pi^{-1}(\Delta) \cup \{\mathrm{exc}(\pi)\}$ is divisor with simple normal crossing (snc) support. Log resolutions exist for all the pairs we will consider in this article, by Hironaka's theorem.

Assume that $K_X + \Delta$ is a $\mathbb{Q}$-Cartier divisor. Given a log resolution of (X, Δ), write

$$\pi^{\star}(K_X + \Delta) = K_Y + \tilde{\Delta} + \sum e_i E_i,$$

where $\tilde{\Delta}$ denotes the proper transform of Δ, and where $\mathrm{exc}(\pi) = \cup E_i$, and E_i are irreducible codimension one subvarieties. Also, assume $\Delta = \sum \delta_i \Delta_i$, with Δ_i irreducible codimension one subvarieties, so $\tilde{\Delta} = \sum \delta_i \tilde{\Delta}_i$. Singularities of pairs can be measured as follows.

Definition 2.1. *Let $Z \subset X$ be a subvariety. A pair (X, Δ) has at most log canonical (lc) singularities (or klt singularities, respectively) along Z if $e_i, \delta_j \leq 1$ for every i (or if $e_i, \delta_j < 1$ for every i, respectively) such that $\pi(E_i) \cap Z \neq \emptyset$ and every j such that $\Delta_j \cap Z \neq \emptyset$.*

On a normal variety, an effective $\mathbb{Q}$-divisor D is a formal linear combination with coefficients in $\mathbb{Q}_+$ of prime divisors. Thus, given such a D and a prime divisor F, one has $D = aF + \Delta$, for some $a \in \mathbb{Q}_+$ and Δ is an effective $\mathbb{Q}$-divisor with $F \not\subset \mathrm{supp}\Delta$.

The number a is called the *vanishing order of D along F*, denoted

$$\mathrm{ord}_F D.$$

2.2 A tool box for complex surfaces

In this chapter, we collect some standard tools that will be useful for us to estimate δ-invariant on complex surfaces (cf. Chapter 4 and 5).

Definition 2.2. *Let D be an effective divisor on S. Suppose that f is the local defining equation of D around the point P, then the multiplicity of D at P, is defined to be the vanishing order of f at P, which we denote by $\mathrm{mult}_P(D)$.*

Remark 2.3. *Let $\pi\colon \widetilde{S} \to S$ be the blow up of the point P, and let E be the exceptional curve of π. Denote by $\widetilde{D}$ the proper transform of D via π. Then we have*

$$\pi^*(D) = \widetilde{D} + \mathrm{mult}_P(D) \cdot E.$$

Definition 2.4. *Let C_1 and C_2 be two irreducible curves on a surface S. Suppose that C_1 and C_2 intersect at P. Let $\mathcal{O}_P$ be the local ring of germs of holomorphic functions defined in some neighborhood of P. Then the local intersection number of C_1 and C_2 at the point P is defined by*

$$\big(C_1 \cdot C_2\big)_P := \dim_{\mathbb{C}} \mathcal{O}_P/(f_1, f_2),$$

where f_1 and f_2 are local defining functions of C_1 and C_2 around the point p. The global intersection number $C_1 \cdot C_2$ is defined by

$$C_1 \cdot C_2 := \sum_{P \in C_1 \cap C_2} \big(C_1 \cdot C_2\big)_P.$$

Note that $C_1 \cdot C_2$ only depends on the numerical classes of C_1 and C_2.

The above two definitions extends to $\mathbb{R}$-divisors by linearity. For instance, say we have a curve C and a $\mathbb{R}$-divisor Δ meeting at the point p. We decompose Δ as $\Delta = \sum_i a_i Z_i$, where Z_i's are distinct prime divisors and $a_i \in \mathbb{R}$. Then,

$$\big(C \cdot \Delta\big)_P := \sum_i a_i \big(C \cdot Z_i\big)_P,$$

where $(C.Z_i)_P = 0$ if Z_i does not pass through the point P.

In the following, let D be an effective $\mathbb{R}$-divisor on S. We will investigate the canonical singularity of the log pair (S, D) at the point P in terms of $\mathrm{mult}_P(\cdot)$ and $\big(\cdot\big)_P$.

Lemma 2.5 ([33]). *If (S, D) is not log canonical at P, then* $\mathrm{mult}_P(D) > 1$.

Let C be an irreducible curve on S. Write

$$D = aC + \Delta,$$

where a is a non-negative real number that is also denoted as $\mathrm{ord}_C(D)$, and Δ is an effective $\mathbb{R}$-divisor on S whose support does not contain the curve C.

Lemma 2.6 ([12, Proposition 3.3]). *Suppose that $a \leqslant 1$, the curve C is smooth at the point P, and* $\mathrm{mult}_P(\Delta) \leqslant 1$. *If (S, D) is not log canonical at P, then*

$$\big(C \cdot \Delta\big)_P > 2 - a.$$

We will give the proof of this lemma in the next section. The following is often referred to as the inversion of adjunction on surfaces.

Corollary 2.7. *(Inversion of adjunction) If $a \leqslant 1$, the curve C is smooth at P, and the log pair (S, D) is not log canonical at P, then*

$$\big(C \cdot \Delta\big)_P > 1.$$

Let $\pi\colon \widetilde{S} \to S$ be the blow up of the point P, and let E_1 be the exceptional curve of π. Denote by $\widetilde{D}$ the proper transform of D via π. Then

$$K_{\widetilde{S}} + \widetilde{D} + \big(\mathrm{mult}_P(D) - 1\big)E_1 \sim_{\mathbb{R}} \pi^*\big(K_S + D\big).$$

This implies

Corollary 2.8. *The log pair (S, D) is log canonical at P if and only if the log pair*

$$\Big(\widetilde{S}, \widetilde{D} + \big(\mathrm{mult}_P(D) - 1\big)E_1\Big)$$

is log canonical along the curve E_1.

Thus, using Lemma 2.5 and Corollary 2.8, we obtain the following simple criterion.

Corollary 2.9. *Suppose that*

$$\mathrm{mult}_Q\big(\pi^*(D)\big) = \mathrm{mult}_P\big(D\big) + \mathrm{mult}_Q\big(\widetilde{D}\big) \leqslant 2$$

for every point $Q \in E_1$. *Then* (S, D) *is log canonical at* P.

If D is a Cartier divisor, then its volume is the number

$$\mathrm{vol}(D) = \limsup_{k\in\mathbb{N}} \frac{h^0(\mathcal{O}_S(kD)}{k^2/2!},$$

where the lim sup can be replaced by a limit (see [36, Example 11.4.7]). Likewise, if D is a $\mathbb{Q}$-divisor, we can define its volume using the identity

$$\mathrm{vol}(D) = \frac{\mathrm{vol}\big(\lambda D\big)}{\lambda^2}$$

for an appropriate $\lambda \in \mathbb{Q}_{>0}$. Then the volume $\mathrm{vol}(D)$ only depends on the numerical equivalence class of the divisor D. Moreover, the volume function can be extended by continuity to $\mathbb{R}$-divisors. Furthermore, it is log-concave:

$$\sqrt{\mathrm{vol}(D_1 + D_2)} \geqslant \sqrt{\mathrm{vol}(D_1)} + \sqrt{\mathrm{vol}(D_2)}. \tag{2.1}$$

for any pseudoeffective $\mathbb{R}$-divisors D_1 and D_2 on the surface S. This fact will be used in our computation in Section 4.1. For more details about volumes of $\mathbb{R}$-divisors, we refer the reader to [35, 36].

If D is not pseudoeffective, then $\mathrm{vol}(D) = 0$. If the divisor D is nef, then

$$\mathrm{vol}(D) = D^2.$$

This follows from the asymptotic Riemann–Roch theorem [36]. If the divisor D is not nef, its volume can be computed using its Zariski decomposition [2, 27, 45, 62]. Namely, if D is pseudoeffective, then there exists a nef $\mathbb{R}$-divisor N on the surface S such that

$$D \sim_{\mathbb{R}} N + \sum_{i=1}^{r} a_i C_i,$$

where each C_i is an irreducible curve on S with $N \cdot C_i = 0$, each a_i is a non-negative real number, and the intersection form of the curves $C_1, \ldots, C_r$ is negative definite. Such decomposition is unique, and it follows from [3, Corollary 3.2] that

$$\mathrm{vol}\big(D\big) = \mathrm{vol}\big(N\big) = N^2.$$

This immediately gives

Corollary 2.10. *Let $Z_1, \ldots, Z_s$ be irreducible curves on S such that $D \cdot Z_i \leqslant 0$ for every i, and the intersection form of the curves $Z_1, \ldots, Z_s$ is negative definite. Then*

$$\mathrm{vol}(D) = \mathrm{vol}\Big(D - \sum_{i=1}^{s} b_i Z_i\Big),$$

where $b_1, \ldots, b_s$ are (uniquely defined) non-negative real numbers such that

$$\Big(D - \sum_{i=1}^{s} b_i Z_i\Big) \cdot Z_j = 0$$

for every j.

Corollary 2.11. *Let Z be an irreducible curve on S such that $Z^2 < 0$ and $D \cdot Z \leqslant 0$. Then*

$$\mathrm{vol}(D) = \mathrm{vol}\Big(D - \frac{D \cdot Z}{Z^2} Z\Big).$$

Let (S, L) be a polarized surface. Let $\eta\colon \widehat{S} \to S$ be a birational morphism (possibly an identity) such that $\widehat{S}$ is smooth. Fix a (non necessarily η-exceptional) irreducible curve F in the surface $\widehat{S}$. Let

$$\tau(F) = \sup\Big\{x \in \mathbb{R}_{>0} \;\Big|\; \eta^*(L) - xF \text{ is pseudoeffective}\Big\}.$$

This is called the pseudoeffective threshold of L with respect to F.

Theorem 2.12. *Suppose that (S, L) is a polarized surface, and $D \sim_{\mathbb{Q}} L$ is a k-basis type divisor with $k \gg 1$. Then*

$$ord_F\big(\eta^*(D)\big) \leqslant \frac{1}{L^2} \int_0^{\tau(F)} \mathrm{vol}\big(\eta^* L - xF\big) dx + \epsilon_k,$$

where ϵ_k is a small constant depending on k such that $\epsilon_k \to 0$ as $k \to \infty$.

Proof: This is a very special case of [26, Lemma 2.2]. □

The following (simple) result can be very handy.

Lemma 2.13. *In the assumptions and notations of Theorem 2.12, one has*

$$\int_{\mu}^{\tau(F)} \mathrm{vol}\big(\eta^* L - xF\big)dx \leqslant \big(\tau(F) - \mu\big)\mathrm{vol}\big(\eta^* L - \mu F\big)$$

for any $\mu \in [0, \tau(F)]$.

Proof: The assertion follows from the fact that $\mathrm{vol}(\eta^* L - xF)$ is a non-increasing function on $x \in [0, \tau(F)]$. □

Using (2.1), this result can be improved as follows:

Lemma 2.14. *In the assumptions and notations of Theorem 2.12, one has*

$$\int_{\mu}^{\tau(F)} \mathrm{vol}\big(\eta^* L - xF\big)dx \leqslant \frac{2}{3}\Big(\tau(F) - \mu\Big)\mathrm{vol}\big(\eta^* L - \mu F\big)$$

for any $\mu \in [0, \tau(F)]$.

Proof: The required assertion follows from the proof of [25, Proposition 2.1]. □

We will apply both Lemmas 2.13 and 2.14 to estimate the integral in Theorem 2.12 in the cases when it is not easy to compute.

2.3 Some new local inequalities on complex surfaces

Let us first give a proof of Lemma 2.6. The proof actually uses Corollary 2.7 (which in turn is a simple application of the inversion of adjunction on surfaces).

Proof of Lemma 2.6. We argue by contradiction. Suppose that

$$(C \cdot \Delta)_P \leq 2 - a.$$

Then we get

$$m := \mathrm{mult}_P(\Delta) \leq 2 - a.$$

Let $\pi : \tilde{S} \to S$ be the blowup of the point P and let E be the exceptional curve of π. Denote by $\tilde{C}$ and $\tilde{\Delta}$ the proper transforms of C and Δ resp. on $\tilde{S}$. Then the log pair

$$\big(\tilde{S}, (a+m-1)E + a\tilde{C} + \tilde{\Delta}\big)$$

is not log canonical at some point $Q \in E$. We claim that this Q has to be the intersection point $E \cap \tilde{C}$. If this is not the case, then the pair

$$\big(\tilde{S}, (a+m-1)E + \tilde{\Delta}\big)$$

is not log canonical at some point $Q \in E$ that is away from $\tilde{C}$. Then Corollary 2.7 applies (as $(a+m-1) \leq 1$) and we obtain

$$m = E \cdot \tilde{\Delta} \geq (E \cdot \tilde{\Delta})_Q > 1,$$

contradicting our assumption that $m \leq 1$. So we see that $Q = E \cap \tilde{C}$. Then applying Corollary 2.7 to the pair $\big(\tilde{S}, (a+m-1)E + a\tilde{C} + \tilde{\Delta}\big)$ at Q gives

$$a - 1 + (C \cdot \Delta)_P = \big(\tilde{C} \cdot ((a+m-1)E + \tilde{\Delta})\big)_Q > 1,$$

and hence

$$(C \cdot \Delta)_P > 2 - a,$$

contradicting our assumption that $(C \cdot \Delta)_P \leq 2 - a$. □

We continue with a new local inequality incorporating also an additional "boundary curve".

Theorem 2.15. *Let S be a surface, let P be a smooth point in S, let Z and C be two irreducible curves on S that both are smooth at P and intersect transversally at P, let $a, b \in [0,1)$ be two non-negative numbers and let Ω be an effective $\mathbb{Q}$-divisor on the surface whose support does not contain the curves C and Z. Suppose that the log pair $(S, (1-b)C + aZ + \Omega)$ is not log canonical at P. Put $m = \mathrm{mult}_P\,\Omega$ and suppose that $m \leq 1$. And also assume that we have either $a + (Z \cdot \Omega)_P - b \leq 1$ or $a + m \leq 1$. Then we have $m > b$ and*

$$(C \cdot \Omega)_P > \frac{m}{m-b}(1-a) - b.$$

Proof: We may assume $b > 0$ (the case when $b = 0$ follows readily from 2.7). We will use an inductive argument.

Let $\pi : \tilde{S} \to S$ be the blowup of the point P and let E be the exceptional curve of π. Denote by $\tilde{C}$, $\tilde{Z}$ and $\tilde{\Omega}$ the proper transforms of C, Z and Δ resp. on $\tilde{S}$. Put

$$\tilde{P} := E \cap \tilde{C},\ \tilde{Q} = E \cap \tilde{Z}.$$

By construction, the log pair

$$\big(\tilde{S}, (1-b)\tilde{C} + a\tilde{Z} + \tilde{\Omega} + (a+m-b)E\big)$$

is not log canonical at some point $O \in E$. Since either $a + (Z \cdot \Omega)_P - b \leq 1$ or $a + m \leq 1$, it is clear that $(a + m - b) \leq 1$.

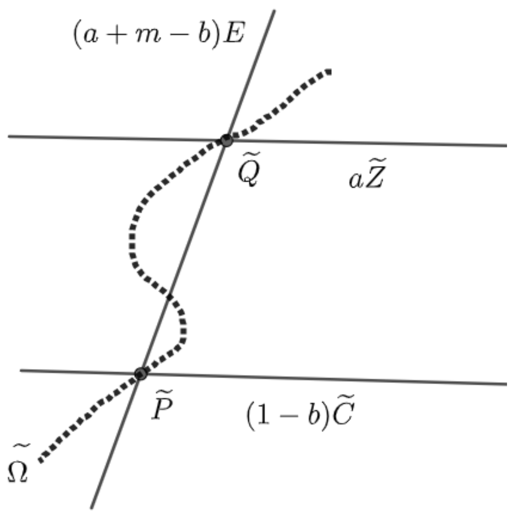

Fig. 2.1: The blowup of P

We first claim that $O = \tilde{P}$ and $m > b$. We argue by contradiction. Suppose that O is away from $\tilde{P}$. Then we claim that $O = \tilde{Q}$. Indeed, if Q is away from both $\tilde{P}$ and $\tilde{Q}$, then the log pair $\big(\tilde{S}, (a+m-b)E + \tilde{\Omega}\big)$ is not log canonical at O so we have

$$m = E \cdot \tilde{\Omega} \geq (E \cdot \tilde{\Omega})_O > 1,$$

contradicting the assumption that $m \leq 1$. So we must have $O = \tilde{Q}$. Then the log pair $\big(\tilde{S}, a\tilde{Z} + \tilde{\Omega} + (a+m-b)E\big)$ is not log canonical at $\tilde{Q}$. We can apply 2.7 w.s.t. both $\tilde{Z}$

and E to derive

$$a - b + (Z \cdot \Omega)_P = \big(\tilde{Z} \cdot (\tilde{\Omega} + (a + m - b)E)\big)_{\tilde{Q}} > 1$$

and

$$a + m \geq \big(E \cdot (a\tilde{Z} + \tilde{\Omega})\big)_{\tilde{Q}} > 1.$$

They contradict the assumption that either $a + (Z \cdot \Omega)_P - b \leq 1$ or $a + m \leq 1$. So we must have $O = \tilde{P}$. Now applying 2.7 to the pair $\big(\tilde{S}, (1-b)\tilde{C} + \tilde{\Omega} + (a + m - b)E\big)$ at $\tilde{P}$, we obtain

$$1 - b + m \geq \big(E \cdot ((1-b)\tilde{C} + \tilde{\Omega})\big)_{\tilde{P}} > 1,$$

and hence

$$m > b,$$

as claimed.

Now we will think of the log pair $(\tilde{S}, (1-b)\tilde{C} + (a + m - b)E + \tilde{\Omega})$ as our *new* pair (as opposed to the original pair $(S, (1-b)C + aZ + \Omega)$), and we put $\tilde{m} := \mathrm{mult}_{\tilde{P}}(\tilde{\Omega})$. Let us check whether this new pair satisfies all the conditions of Theorem 2.15.

Suppose that we have both

$$(a + m - b) + (E \cdot \tilde{\Omega})_{\tilde{P}} - b > 1$$

and

$$(a + m - b) + \tilde{m} > 1.$$

Then

$$(C \cdot \Omega)_P = m + (\tilde{C} \cdot \tilde{\Omega})_{\tilde{P}} \geq m + \tilde{m} > 1 - a + b.$$

Thus

$$2 < \frac{(C \cdot \Omega)_P - (1-a) + b}{b}.$$

Meanwhile, the inequality $(a + m - b) + (E \cdot \tilde{\Omega})_{\tilde{P}} - b > 1$ gives

$$2 > \frac{1-a}{m-b}.$$

So we derive

$$\frac{(C \cdot \Omega)_P - (1-a) + b}{b} > \frac{1-a}{m-b},$$

and hence

$$(C \cdot \Omega)_P > \frac{m}{m-b}(1-a) - b.$$

Then we are done.

Thus we can assume that either $(a+m-b)+(E\cdot\tilde{\Omega})_{\tilde{P}}-b \leq 1$ or $(a+m-b)+\tilde{m} \leq 1$. This forces $(a+m-b) < 1$ (otherwise our new log pair $(\tilde{S}, (1-b)\tilde{C}+(a+m-b)E+\tilde{\Omega})$ would be log canonical at $\tilde{P}$). So the new pair satisfies all the conditions in Theorem 2.15. One can now blow up $\tilde{P}$ and repeat the previous argument for our new pair. But observe that, the intersection number $(\tilde{C} \cdot \tilde{\Omega})_{\tilde{P}}$, when compared to the corresponding intersection number $(C \cdot \Omega)_P$ of the previous pair, strictly decreases by $m \geq b > 0$. So this blowup argument cannot be repeated infinitely times and the conclusion of Theorem 2.15 has to be true at certain stage after repeating the previous argument sufficiently many times. By induction, we might as well assume that the theorem already holds for $(\tilde{S}, (1-b)\tilde{C} + (a+m-b)E + \tilde{\Omega})$. Namely, we have $\tilde{m} > b$ and

$$(\tilde{C} \cdot \tilde{\Omega})_{\tilde{P}} > \frac{\tilde{m}}{\tilde{m}-b}(1-(a+m-b)) - b.$$

Or equivalently,

$$(C \cdot \Omega)_P - m > \frac{\tilde{m}}{\tilde{m}-b}(1-(a+m-b)) - b.$$

Using the fact $\tilde{m} \leq m$, we derive

$$(C \cdot \Omega)_P - m > \frac{m}{m-b}(1-(a+m-b)) - b,$$

and hence

$$(C \cdot \Omega)_P > \frac{m}{m-b}(1-a) = b.$$

This completes the proof.

□

Theorem 2.16. *Let S be a surface, let p be a smooth point in S, let Z and C be two irreducible curves on S that both are smooth at p and intersect transversally at p, let b be a non-negative number such that $b \leq 1$, let a be a non-negative number such that $a \leq 1$, and let Ω be an effective $\mathbb{Q}$-divisor on the surface whose support does not contain the* *curves C and Z. Suppose that the log pair $(S, (1-b)C + aZ + \Omega)$ is not log canonical at p. Put $m = mult_p\Omega$ and suppose that $m \leq 1$. Then we have*

$$(C \cdot \Omega)_p > \frac{(Z \cdot \Omega)_p}{(Z \cdot \Omega)_p - b}(1-a) - b.$$

Proof: We may assume that $b > 0$. If we have either $a + (Z \cdot \Omega)_p - b \leq 1$ or $a + m \leq 1$, then by Lemma 2.6, it is easy to check that $a < 1$ and $b < 1$. So we can apply Theorem 2.15 and the result follows immediately since we have $m \leq (Z \cdot \Omega)_p$.

So we may assume that we have both

$$a + (Z \cdot \Omega)_p - b > 1 \text{ and } a + m > 1.$$

Then we get

$$(C \cdot \Omega)_p \geq m > 1 - a.$$

So we have

$$b < (C \cdot \Omega)_p - (1-a) + b,$$

so that

$$1 < \frac{(C \cdot \Omega)_p - (1-a) + b}{b}.$$

In the meantime, the inequality $a + (Z \cdot \Omega)_p - b > 1$ gives

$$1 > \frac{1-a}{(Z \cdot \Omega)_p - b}.$$

So we get

$$\frac{(C \cdot \Omega)_p - (1-a) + b}{b} > \frac{1-a}{(Z \cdot \Omega)_p - b},$$

so that

$$(C \cdot \Omega)_p > \frac{(Z \cdot \Omega)_p}{(Z \cdot \Omega)_p - b}(1-a) - b.$$

The proof is complete. □

The following estimate gives a better bound than the usual inversion of adjunction.

Theorem 2.17. *Let S be a surface, let p be a smooth point in S, let C be an irreducible curve on S that is smooth at p, let b be a non-negative number such that $b < 1$, and let Ω be an effective $\mathbb{Q}$-divisor on the surface whose support does not contain the curve C.*

Suppose that the log pair $(S, (1-b)C + \Omega)$ is not log canonical at p. Put $m = mult_p\Omega$ and suppose that $m \le 1$. Then we have

$$(C \cdot \Omega)_p > 1 + \frac{b^2 + (1-m)b}{m-b}.$$

Proof: Locally we may pick a general curve Z passing through p such that Z is smooth at p and it intersects transversally with C. Then we choose $a = 0$. Notice that now we can apply Theorem 2.15 to the pair $(S, (1-b)C + aZ + \Omega)$. So we get

$$(C \cdot \Omega)_p > \frac{m}{m-b} - b = 1 + \frac{b^2 + (1-m)b}{m-b}.$$

□

Theorem 2.18. *Let S be a surface, let p be a smooth point in S, let Z be an irreducible curve on S that is smooth at p, let a be a positive number such that $a \le 1$, and let Ω be an effective $\mathbb{Q}$-divisor on the surface whose support does not contain the curve C. Suppose that the log pair $(S, aZ + \Omega)$ is not log canonical at p. Put $m = mult_p\Omega$ and suppose that $m \le 1$. Then we have*

$$(Z \cdot \Omega)_p > 1 + \frac{1-a}{m+a}.$$

Proof: Locally we may pick a general curve C passing through p such that C is smooth at p and C intersects transversally with Z at p. Moreover we may assume that

$$m = (C \cdot \Omega)_p.$$

We then choose $b = 1$ and apply Theorem 2.16. So we get

$$m > \frac{(Z \cdot \Omega)_p}{(Z \cdot \Omega)_p - 1}(1-a) - 1.$$

Then we have

$$(Z \cdot \Omega)_p > 1 + \frac{1-a}{m+a}.$$

□

2.4 Tian's alpha invariant

As we mentioned in the Introduction 1, Tian's α-invariant plays an important role in the study of Kähler-Einstein problem on Fano manifolds. The purpose of this section is to give some general properties of the α-invaraint. Let us start with the definition.

Definition 2.19. *Let (X,ω) be a compact Kähler manifold. We define*

$$\alpha(X,[\omega]) = \sup\{\alpha > 0 \mid \exists C_\alpha > 0 \ s.t. \int_X e^{-\alpha(\phi - \sup_X \phi)}\omega^n \leq C_\alpha, \ \forall \phi \in \mathcal{H}(X,\omega)\}.$$

Tian [52] proved that, for any compact Kähler manifold, such an invariant must be a positive number. It is direct to check that this invariant only depends on the Kähler class $[\omega]$, so the notation $\alpha(X,[\omega])$ makes sense, which we will call the α-invariant of (X,ω).

In the following, we will always work with polarized Kähler manifolds, namely, there is also an ample line bundle L on X and $\omega \in 2\pi c_1(L)$. In this case, we shall write

$$\alpha(X,L) := \alpha(X,[\omega]).$$

When $L = -K_X$, we will also write $\alpha(X) := \alpha(X,-K_X)$ for simplicity, which will be called the α-invariant of X. Let us also fix a smooth Hermitian metric h on L.

Definition 2.20 ([53])**.** *For each $m \geq 1$, we define*

$$\alpha_m(X,L) = \sup\{\lambda > 0 \mid \int_X |s|_{h^m}^{-2\lambda/m}\omega^n < \infty, \ \mathit{for} \ \forall s \in H^0(X,L^m), s \neq 0\}$$

$\alpha_m(X,L)$ can be thought of as a finite dimensional quantization of $\alpha(X,L)$, and we have the following

Theorem 2.21. *([13], [49]) We have*

$$\alpha(X,L) = \inf_m \alpha_m(X,L) = \lim_{m\to+\infty} \alpha_m(X,L).$$

From the view of modern algebraic geometry, the quantity $\alpha_m(X,L)$ can also be related to the log canonical threshold of the pair (X,L^m), denoted $\mathrm{lct}_m(X,L)$. Here

$$\mathrm{lct}_m(X,L) := m\sup\{\lambda \mid (X,\lambda D) \text{ is log canonical for any effective } \ D \in |L^m|\}$$

And we have the following relation

$$\alpha_m(X, L) = \mathrm{lct}_m(X, L).$$

(This can be seen from the fact that the log canonicity of an effective divisor is equivalent to certain integrability of the local holomorphic defining function of the divisor.) In particular, Theorem 2.21 gives the following purely algebraic definition of α invariants.

$$\alpha(X, L) = \sup\left\{\lambda > 0 \,\middle|\, \begin{array}{l}\text{the log pair } (X, \lambda D) \text{ is log canonical} \\ \text{for every effective } \mathbb{Q}\text{-divisor } D \sim_{\mathbb{Q}} L\end{array}\right\}. \tag{2.2}$$

This algebraic characterization is easier to work with if one wants to compute the α-invariant explicitly.

Example 2.22. *The α-invariant of the projective plane $\mathbb{P}^2$ is*

$$\alpha(\mathbb{P}^2, -K_{\mathbb{P}^2}) = \frac{1}{3}.$$

Proof: Note that $-K_{\mathbb{P}^2} = \mathcal{O}(3)$. So if we pick an arbitrary line L on $\mathbb{P}^2$, then

$$3L \sim -K_{\mathbb{P}^2}.$$

Since $(\mathbb{P}^2, 3\lambda L)$ is log canonical if and only if $\lambda \leq 1/3$, we obtain from (2.2) that

$$\alpha(\mathbb{P}^2, -K_{\mathbb{P}^2}) \leq \frac{1}{3}.$$

To show the equality, we argue by contradiction. Suppose that there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_{\mathbb{P}^2}$ such that the log pair $(\mathbb{P}^2, \frac{1}{3}D)$ is not log canonical at some point $P \in \mathbb{P}^2$. Let us pick a general line L passing through P, which is not contained in the support of D. Then the log pair $(\mathbb{P}^2, L + \frac{1}{3}D)$ is also not log canonical at P. Applying Corollary 2.7 to this pair, we derive

$$1 = \big(\mathcal{O}(1)\big)^2 = L \cdot (\frac{1}{3}D) \geq \big(L \cdot (\frac{1}{3}D)\big)_P > 1,$$

a contradiction. □

Example 2.23. *Let S be a smooth Fano surfaces (i.e. $\dim S = 2$ and $-K_S$ is ample).*

All possible values of $\alpha(S,-K_S)$ have been computed in [11].

$$\alpha(S,-K_S)=\begin{cases}\frac{1}{3} \text{ if } S\cong\mathbb{F}_1 \text{ or } K_S^2\in\{7,9\},\\ \frac{1}{2} \text{ if } S\cong\mathbb{P}^1\times\mathbb{P}^1 \text{ or } K_S^2\in\{5,6\},\\ \frac{2}{3} \text{ if } K_S^2=4,\\ \frac{2}{3} \text{ if } S \text{ is a cubic surface in } \mathbb{P}^3 \text{ with an Eckardt point},\\ \frac{3}{4} \text{ if } S \text{ is a cubic surface in } \mathbb{P}^3 \text{ without Eckardt points},\\ \frac{3}{4} \text{ if } K_S^2=2 \text{ and } |-K_S| \text{ has a tacnodal curve},\\ \frac{5}{6} \text{ if } K_S^2=2 \text{ and } |-K_S| \text{ has no tacnodal curves},\\ \frac{5}{6} \text{ if } K_S^2=1 \text{ and } |-K_S| \text{ has a cuspidal curve},\\ 1 \text{ if } K_S^2=1 \text{ and } |-K_S| \text{ has no cuspidal curves}.\end{cases}$$

Chapter 3 Stability thresholds on Fano type manifolds

3.1 Fujita-Odaka's invaraint

Let (X, L) be a polarized pair, where X is an n-dimensional projective manifold and L is an ample line bundle on X. Recently, Fujita-Odaka [26] introduced an δ-invariant in the study of K-stability of Fano varieties. We begin with a general definition.

Definition 3.1. *For any $k \geq 1$, we set*

$$d_k := dim_{\mathbb{C}} H^0(X, L^k) > 0.$$

For any basis $s_1, ..., s_{d_k}$ of $H^0(X, kL)$, let D_i be the divisor cut out by s_i and we consider the $\mathbb{Q}$-divisor

$$D = \frac{1}{kd_k} \sum_{i=0}^{d_k} D_i,$$

which we call a k-basis type divisor. We set

$$\delta_k(X, L) := \{c > 0 |\ (X, cD) \text{ is lc for any } k\text{-basis type divisor } D\}.$$

And we define the delta invariant by

$$\delta(X, L) := \limsup_{k \to \infty} \delta_k(X, L).$$

(We remark that the limsup is actually a limit; see [4, Theorem A].) If $L = -K_X$, then we simply write

$$\delta(X) := \delta(X, -K_X),$$

which is called the δ-invariant (or the stability threshold) of X.

It turns out that $\alpha(X, L)$ and $\delta(X, L)$ have the following nice relation (cf. [4])

$$\frac{n+1}{n}\alpha(X, L) \leq \delta(X, L) \leq (n+1)\alpha(X, L). \tag{3.1}$$

Both α-invariant and δ-invariant are particularly useful in the Fano setting, i.e. when $L = -K_X$. In this case, these two invariants are closely related to the existence of KE metrics and K-stability; see Theorem 1.1 and 1.2. For instance, when $\alpha(X) > \frac{n}{n+1}$, (3.1) implies that $\delta(X) > 1$, so Theorem 1.2 shows that X is uniformly K-stable. On the other hand, suppose that X is K-semistable, namely $\delta(X) \geq 1$. Then (3.1) implies that one must have $\alpha(X) \geq \frac{1}{n+1}$, i.e. the α-invariant of a K-semistable Fano manifold cannot be too small.

3.2 Delta invaraint and the greatest Ricci lower bound

Throughout this section, X will be an n-dimensional Fano manifold. The purpose of this section is to relate the δ-invaraint of X to an analytic quantity called the greatest Ricci lower bound. We begin with the following definition.

Definition 3.2 ([46, 47, 56]). *We define the greatest Ricci lower bound $\beta(X)$ to be*

$$\beta(X) := \sup\{\lambda > 0 \mid \exists\ \omega \in 2\pi c_1(X) \text{ such that } Ric(\omega) > \lambda\omega\ \}.$$

This invariant was the topic of Tian's article [56] although it was not explicitly defined there, but was first explicitly defined by Y. Rubinstein in [46, (32)], [47, Problem 3.1] and was later further studied by Székelyhidi [51], Li [37], Song–Wang [50], and Cable [10]. Roughly speaking, $\beta(X)$ measures how far X is from being a Kähler-Einstein (KE) manifold. So it is always an interesting problem to find the value of $\beta(X)$ since it plays an important role in the study of KE problems.

Remark 3.3. *The threshold $\beta(X)$ is also closely related to the alpha invariant $\alpha(X)$. For instance, we have*

$$\beta(X) \geqslant \min\big\{\frac{n+1}{n}\alpha(X), 1\big\},$$

which can be derived using the continuity method; see [52] and also [47, Lemma 6.2].

As conjectured by Rubinstein [46, Problem 4.1], both $\beta(X)$ and $\delta(X)$ can be used to test K-(semi)stability of X (for the definition of K-(semi)stability, we refer the reader to [5]). And indeed, by the work of many people, we now have the following result.

Theorem 3.4 ([4, 26, 38]). *The following are equivalent.*

1. *X is K-semistable;*

2. $\beta(X) = 1$;
3. $\delta(X) \geqslant 1$.

The main result of this section is the following

Theorem 3.5. *Let X be a Fano manifold. Then we have*

$$\beta(X) = \min\{\delta(X), 1\}.$$

This result can be thought of as a special version of the YTD correspondence, since it relates an analytic quantity to an algebraic quantity. Note that this result was first proved in the toric case by Blum-Jonsson [4, Corollary 7.19]. So it is reasonable to believe that the same result holds in the general setting. The purpose of this section is to give a short proof of Theorem 3.5 relying on some recent developments in the literature. (Note this result has also been proved independently in the recent work [7])

For the purpose of the proof, we generalize the definition of δ-invariant to $\mathbb{Q}$-line bundles.

Definition 3.6 ([4, 26])**.** *Let L be an ample $\mathbb{Q}$-line bundle on X. For any sufficiently large and divisible integer k, we consider a basis $s_1, \cdots s_{d_k}$ of $H^0(X, kL)$, where $d_k = h^0(X, kL)$. We can associate a $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} L$ to this basis by*

$$D := \frac{1}{kd_k} \sum_{i=1}^{d_k} \{s_i = 0\}.$$

Any D obtained in this way is called a k-basis type divisor of L. We put

$$\delta_k(L) := \sup\{c > 0 \mid (X, cD) \text{ is log canonical for any } k\text{-basis type divisor } D \text{ of } L\}.$$

Then we define $\delta(L)$ by

$$\delta(L) := \limsup_k \delta_k(L).$$

To prove Theorem 3.5, one also needs to use Kähler-Einstein edge (KEE) metric and its corresponding thresholds as well. So we recall the following two definitions.

Definition 3.7 ([38])**.** *Suppose that $\Delta \in |-mK_X|$ is a smooth divisor, where m is a positive integer. We define*

$$\beta(X, \Delta/m) := \sup\{\lambda > 0 \mid \exists \text{ KEE metric } \omega \in 2\pi c_1(X) \text{ s.t. } Ric(\omega) = \lambda\omega + 2\pi(1-\lambda)[\Delta]/m\}$$

Definition 3.8 ([18]). *Suppose that $\Delta \in |-mK_X|$ is a smooth divisor, where m is a positive integer. Let $\lambda \in (0,1]$ be a rational number. Then $-K_X - \frac{1-\lambda}{m}\Delta \sim_{\mathbb{Q}} -\lambda K_X$ is an ample $\mathbb{Q}$-line bundle. We define*

$$\delta_k(X, \frac{1-\lambda}{m}\Delta) := \sup\left\{ c>0 \;\middle|\; \begin{aligned} &\text{the log pair } \left(X, \frac{1-\lambda}{m}\Delta + cD\right) \text{ is log canonical} \\ &\text{for any } k\text{-basis type divisor } D \text{ of } -\lambda K_X \end{aligned} \right\}.$$

Moreover, we define

$$\delta(X, \frac{1-\lambda}{m}\Delta) := \limsup_k \delta_k(X, \frac{1-\lambda}{m}\Delta),$$

which is called the δ-invariant of the log Fano pair $(X, \frac{1-\lambda}{m}\Delta)$.

For more information about KE and KEE metrics, we refer to [48]. Note that, by Bertini's theorem, for $m \gg 1$, any general divisor $\Delta \in |-mK_X|$ is smooth.

Remark 3.9. *$\beta(X)$ and $\beta(X, \Delta/m)$ can be related as follows:*

$$\beta(X)\frac{m-1}{m-\beta(X)} \leqslant \beta(X, \Delta/m) \leqslant \beta(X). \tag{3.2}$$

See [50] for a proof (see also [38]). In particular, $\lim \beta(X, \Delta/m) = \beta(X)$.

The thresholds $\beta(X, \Delta/m)$ and $\delta(X, \frac{1-\lambda}{m}\Delta)$ are the counterparts of $\beta(X)$ and $\delta(X)$ in the log setting. They can be used to test the existence of KEE metrics and log K-(semi)stability (see e.g. [18,40]).

If follows immediately from the definition that

$$\delta(-\lambda K_X) \geqslant \delta(X, \frac{1-\lambda}{m}\Delta). \tag{3.3}$$

With a little more effort, one can actually prove the following

Lemma 3.10. *Fix a rational number $\lambda \in (0,1]$. For each $m \gg 1$ pick a smooth divisor $\Delta \in |-mK_X|$ and put $B_m := \frac{1-\lambda}{m}\Delta$. Then we have*

$$\lim_{m\to\infty} \delta(X, B_m) = \delta(-\lambda K_X).$$

Proof: Fix any small $\epsilon > 0$. If suffices to show that, for any $m \gg 1$ and sufficiently divisible $k \gg 1$, we have

$$\delta_k(-\lambda K_X) \geqslant \delta_k(X, B_m) \geqslant (1-\epsilon)\delta_k(-\lambda K_X).$$

The first inequality $\delta_k(-\lambda K_X) \geqslant \delta_k(X, B_m)$ follows immediately from Definition 3.6 and Definition 3.8. So it remains to prove the second inequality. For this purpose, we let D be any k-basis type divisor of $-\lambda K_X$. Pick any $c > 0$ such that the log pair (X, cD) is log canonical. If then suffices to show that the log pair $(X, B_m + (1-\epsilon)cD)$ is log canonical as well.

Now notice that, the log pair (X, Δ) is log canonical since $\Delta \in |-mK_X|$ is a *smooth* divisor. Then we can apply a trick from [11] to show the log canonicity of $(X, B_m + (1-\epsilon)cD)$. Indeed, suppose that the log pair $(X, B_m + (1-\epsilon)cD)$ is not log canonical, then [11, Remark 2.1] implies that the log pair $(X, \frac{(1-\epsilon)c}{1-(1-\lambda)/m}D)$ is not log canonical as well. If we pick $m \geqslant \frac{1-\lambda}{\epsilon}$, the log pair (X, cD) is then not log canonical, contradicting our choice of c. □

Now we are ready to prove Theorem 3.5.

Proof of Theorem 3.5. Using Theorem 3.4, it is enough to assume that X is not K-semistable. So $\beta(X) \in (0,1)$. Our goal is to show that $\delta(X) = \beta(X)$. For simplicity we may also assume that $\beta(X) \in \mathbb{Q}$. Then we consider the ample $\mathbb{Q}$-line bundle $-\beta(X)K_X$. By Definition 3.6, it suffices to show that

$$\delta(-\beta(X)K_X) = 1.$$

First, we show that $\delta(-\beta(X)K_X) \geqslant 1$. For this purpose, we pick any rational number $\lambda \in (0, \beta(X))$. Then let m be a sufficiently large integer and pick a smooth divisor $\Delta \in |-mK_X|$. By (3.2), we may assume that

$$\lambda < \beta(X, \Delta/m).$$

Then by Definition 3.7 and [40, Theorem 1.1], we can find a KEE metric $\omega \in 2\pi c_1(X)$ such that

$$Ric(\omega) = \lambda\omega + 2\pi(1-\lambda)[\Delta]/m.$$

So the log pair $(X, \frac{\lambda-1}{m}\Delta)$ is log K-semistable (see [40, Corollary 1.12]). Thus by [18, Corollary 4.8], we have

$$\delta(X, \frac{\lambda-1}{m}\Delta) \geqslant 1.$$

Hence by (3.3), we have $\delta(-\lambda K_X) \geqslant 1$. Letting $\lambda \to \beta(X)$, we get

$$\delta(-\beta(X)K_X) \geqslant 1.$$

So it remains to show that $\delta(-\beta(X)K_X) \leqslant 1$. We argue by contradiction. Suppose that $\delta(-\beta(X)K_X) > 1$. Then we may pick a sufficiently small rational number $\epsilon > 0$ such that $\beta(X) + \epsilon \leqslant 1$ (recall that $\beta(X) < 1$) and

$$\delta\big(-(\beta(X)+\epsilon)K_X\big) > 1.$$

Then Lemma 3.10 implies that, for any $m \gg 1$ and any smooth divisor $\Delta \in |-mK_X|$, we have

$$\delta(X, \frac{1-(\beta(X)+\epsilon)}{m}\Delta) > 1.$$

Then by [8, Corollary 2.11], the log pair $\big(X, \frac{1-(\beta(X)+\epsilon)}{m}\Delta\big)$ is uniformly log K-stable. So it follows from [17, 58] (see also [59]) that, there exists a KEE metric associated to this pair. Thus we have $\beta(X, \Delta/m) \geqslant \beta(X) + \epsilon$, contradicting (3.2). □

Remark 3.11. *In the above argument, to prove $\delta(-\beta(X)K_X) \leqslant 1$, one can also argue as follows. Suppose that $\delta(-\beta(X)K_X) > 1$. Then we may pick a sufficiently small rational number $\epsilon > 0$ such that $\beta(X) + \epsilon \leqslant 1$ and $\delta\big(-(\beta(X)+\epsilon)K_X\big) \geqslant 1$. Then it follows from [8, Corollary 2.11] that, the polarized pair $\big(X, -(\beta(X)+\epsilon)K_X\big)$ is K-semistable in the adjoint sense, hence twisted K-semistable in the sense of [21] (see [5, Proposition 8.2]). So [20, Proposition 10] guarantees that, for some $\lambda \in (\beta(X), \beta(X)+\epsilon)$, we can find two Kähler forms ω,$\alpha \in 2\pi c_1(X)$ such that*

$$Ric(\omega) = \lambda\omega + (1-\lambda)\alpha,$$

which also gives us a contradiction.

Now we show that, Theorem 3.5 has the following consequence.

Theorem 3.12. *Let X and Y be two smooth Fano manifolds. Then we have*

$$\beta(X \times Y) = \min\{\beta(X), \beta(Y)\}.$$

Note that this can be proved using analytic methods as well, since $\beta(X)$ corresponds to the maximal existence time of the solution to the continuity method (cf. [51]). But here we present a short algebraic proof with the help of the δ-invaraint.

Proof: By definition 3.2, it is clear that

$$\beta(X \times Y) \geq \min\{\beta(X), \beta(Y)\}.$$

If $\beta(X) = \beta(Y) = 1$, then we must have $\beta(X \times Y) = 1$, so we are done. Thus we may assume that $\beta(X) \leq \beta(Y)$ and that $\beta(X) < 1$. So in particular, $\beta(X) = \delta(X)$ (recall Theorem 3.5).

On the other hand, by Definition 3.6, it is easy to check that

$$\delta(X \times Y) \leq \min\{\delta(X), \delta(Y)\}.$$

Now using Theorem 3.5 we derive

$$\beta(X \times Y) = \delta(X \times Y) \leq \min\{\delta(X), \delta(Y)\} = \min\{\beta(X), \beta(Y)\}.$$

This completes the proof. □

Remark 3.13. *For α-invariant, it was known that ([13, Lemma 2.29])*

$$\alpha(X \times Y) = \min\{\alpha(X), \alpha(Y)\}$$

For δ-invariant, Park-Won conjectured that ([43, Conjecture 1.11])

$$\delta(X \times Y) = \min\{\delta(X), \delta(Y)\}.$$

Our Theorem 3.12 shows that this is indeed true if $\delta(X)$ or $\delta(Y)$ is no bigger than one. The author was recently informed that this conjecture has now been fully resolved by Ziquan Zhuang [63].

3.3 Analytic delta invariant

Let X be a smooth Fano variety of dimension n. Suppose that $\omega \in 2\pi c_1(X)$ is a Kähler metric. We put

$$\mathcal{H}(X,\omega) = \{\phi \in C^\infty(X,\mathbb{R}) \mid \omega_\phi = \omega + \sqrt{-1}\partial\bar{\partial}\phi > 0\}.$$

This is called the space of Kähler potentials of (X,ω).

Let us introduce several useful functionals on $\mathcal{H}(X,\omega)$. Put $V = \int_X \omega^n$.

The I-functional $I_\omega(\cdot)$ is defined by

$$I_\omega(\phi) := \frac{1}{V}\sum_{i=0}^{n-1}\int_X \sqrt{-1}\partial\phi \wedge \bar{\partial}\phi \wedge \omega^i \wedge \omega_\phi^{n-1-i};$$

The J-functional $J_\omega(\cdot)$ is defined by

$$J_\omega(\phi) := \frac{1}{V}\sum_{i=0}^{n-1}\frac{i+1}{n+1}\int_X \sqrt{-1}\partial\phi \wedge \bar{\partial}\phi \wedge \omega^i \wedge \omega_\phi^{n-1-i};$$

The Mabuchi K-energy $\mathcal{M}_\omega(\cdot)$ is defined by

$$\mathcal{M}_\omega(\phi) := \frac{1}{V}\int_X \log\frac{\omega_\phi^n}{\omega^n}\omega_\phi^n + \frac{1}{V}\int_X h_\omega(\omega - \omega_\phi^n) - (I_\omega - J_\omega)(\phi).$$

These functionals are important in the study of canonical metrics and they are used to derive a priori estimates for potential functions along the continuity method (or along the Kähler-Ricci flow). We refer to the survey [48] for more details.

We recall that, in [51], the greatest Ricci lower bound $\beta(X)$ is related to certain properness of the K-energy. Inspired by this and by Theorem 3.5, we introduce an analytic delta invariant.

Definition 3.14. *The analytic delta invariant $\tilde{\delta}(X)$ is defined by*

$$\tilde{\delta}(X) := \sup\{\delta > 0 \mid \exists C_\delta > 0 \ s.t.\ \mathcal{M}_\omega \geq (\delta - 1)(I_\omega - J_\omega) - C_\delta\}.$$

The Mabuchi K-energy is said to be proper if $\tilde{\delta}(X) > 1$, in which case, one can derive uniform C^0 estimate along the continuity method (or along the Kähler-Ricci flow) to show the existence of KE metrics. In some sense, the converse is also true.

Theorem 3.15. *[44, 55] Suppose that X does not admit non-trivial holomorphic vector field. Then X admits a KE metric if and only if $\tilde{\delta}(X) > 1$.*

This can be thought of as an analytic version of the YTD correspondence.

We conjecture that $\tilde{\delta}(X) = \delta(X)$. When $\tilde{\delta}(X) \leq 1$, this is indeed true, since both $\tilde{\delta}(X)$ and $\delta(X)$ coincide with the greatest Ricci lower bound $\beta(X)$ (cf. Theorem 3.5 and [51]). Moreover we remark that, in the recent work [8], it was proved that $\delta(X) \geq \tilde{\delta}(X)$ using non-Archimedean approach. It seems to the author that the reverse direction is still missing.

Chapter 4 Delta invariants on smooth cubic surfaces

In [43], Park and Won estimated δ-invariants of all smooth del Pezzo surfaces, which gives a purely algebraic proof of Tian's work [53] when combined with Theorem 1.2. In this chapter, we give an alternative and more geometric approach to the same problem. For simplicity we only focus on smooth cubic surfaces, since this is the hardest case in to deal with in [53].

Our main result is the following

Theorem 4.1. *Let S be a smooth cubic surface in $\mathbb{P}^3$. Then $\delta(S) \geqslant \frac{6}{5}$.*

So Theorem 1.2 immediately gives

Corollary 4.2 ([43, 53])**.** *All smooth cubic surfaces in $\mathbb{P}^3$ are uniformly K-stable.*

For a smooth cubic surface S, it was proved in [43, Theorem 4.9] that

$$\delta(S) \geqslant \frac{36}{31}.$$

The proof of Theorem 4.1 is completely different from the proof of [43, Theorem 4.9]. Moreover, our bound $\delta(S) \geqslant \frac{6}{5}$ is slightly better.

4.1 Multiplicity estimates

Let S be a smooth cubic surface in $\mathbb{P}^3$, and let D be a k-basis type divisor with $k \gg 1$. The goal of this section is to bound multiplicities of the divisor D using Theorem 2.12. As in Theorem 2.12, we denote by ϵ_k a small number such that $\epsilon_k \to 0$ as $k \to \infty$.

Lemma 4.3. *Let L be a line on S. Then*

$$ord_L(D) \leqslant \frac{5}{9} + \epsilon_k.$$

Proof: Let us use assumptions and notations of Theorem 2.12 with $\eta = \mathrm{Id}_S$ and $F = L$. Let H be a general hyperplane section of the surface S that contains L. Then $H = L+C$, where C is an irreducible conic. Since $C^2 = 0$, we have $\tau(F) = 1$, so that

$$\mathrm{ord}_L(D) \leqslant \frac{1}{3}\int_0^1 \mathrm{vol}(-K_S - xL)dx + \epsilon_k = \frac{1}{3}\int_0^1 (-K_S - xL)^2 dx + \epsilon_k = \frac{5}{9} + \epsilon_k$$

by Theorem 2.12. □

Fix a point $P \in S$. Let $\pi\colon \widetilde{S} \to S$ be the blowup of this point. Denote by E_1 the exceptional divisor of π. Fix a point $Q \in E_1$. Let $\sigma\colon \widehat{S} \to \widetilde{S}$ be the blowup of this point. Denote by E_2 the exceptional curve of σ. Let $\eta = \pi \circ \sigma$ and $F = E_2$. Let

$$\tau(E_2) = \sup\Big\{x \in \mathbb{R}_{>0} \ \Big|\ \eta^*(-K_S) - xF \text{ is pseudoeffective}\Big\}.$$

Applying Theorem 2.12, we get

$$\mathrm{mult}_Q\big(\pi^*(D)\big) \leqslant \frac{1}{3}\int_0^{\tau(E_2)} \mathrm{vol}\big(\eta^*(-K_S) - xE_2\big)dx + \epsilon_k. \qquad (4.1)$$

Let T_P be the unique hyperplane section of the surface S that is singular at the point P. Then we have the following four possibilities:

- $T_P = L_1 + L_2 + L_3$, where L_1, L_2 and L_3 are lines such that $P = L_1 \cap L_2 \cap L_3$;
- $T_P = L_1 + L_2 + L_3$, where L_1, L_2 and L_3 are lines such that $L_3 \not\ni P = L_1 \cap L_2$;
- $T_P = L + C$, where L is a line and C is a conic such that $P \in C \cap L$.
- T_P is an irreducible cubic curve.

We plan to bound the integral in (4.1) depending on the type of the curve T_P and on the position of the point $Q \in E_1$. First, we deal with the cases when Q is contained in the proper transform of the curve T_P. We start with

Lemma 4.4. *Suppose that $T_P = L_1 + L_2 + L_3$, where L_1, L_2 and L_3 are lines passing through P. Let $\widetilde{L}_1$, $\widetilde{L}_2$ and $\widetilde{L}_3$ be the proper transforms on $\widetilde{S}$ of the lines L_1, L_2 and L_3, respectively. Suppose that $Q \in \widetilde{L}_1 \cap \widetilde{L}_2 \cap \widetilde{L}_3$. Then*

$$\mathrm{mult}_Q\big(\pi^*(D)\big) \leqslant \frac{17}{9} + \epsilon_k.$$

Proof: We may assume that $Q = \widetilde{L}_1 \cap E_1$. Denote by $\widehat{L}_1$, $\widehat{L}_2$, $\widehat{L}_3$ and $\widehat{E}_1$ the proper transforms on $\widehat{S}$ of the curves $\widetilde{L}_1$, $\widetilde{L}_2$, $\widetilde{L}_3$ and E_1, respectively. Then the intersection form of the curves $\widehat{L}_1$, $\widehat{L}_2$, $\widehat{L}_3$ and $\widehat{E}_1$ is negative definite. Moreover, we have

$$\eta^*(-K_S) \sim_{\mathbb{Q}} \widehat{L}_1 + \widehat{L}_2 + \widehat{L}_3 + 3\widehat{E}_1 + 4E_2.$$

Thus, we conclude that $\tau(E_2) = 4$. Now, using Corollary 2.10, we compute

$$\mathrm{vol}\big(\eta^*(-K_S) - xE_2\big) = \begin{cases} 3 - \frac{x^2}{2}, & 0 \leqslant x \leqslant 1, \\ \frac{20-4x-x^2}{6}, & 1 \leqslant x \leqslant 2, \\ \frac{(4-x)^2}{3}, & 2 \leqslant x \leqslant 4. \end{cases}$$

Then the required result follows from (4.1). □

Lemma 4.5. *Suppose that $T_P = L_1 + L_2 + L_3$, where L_1, L_2 and L_3 are lines such that $P = L_1 \cap L_2$ and $P \notin L_3$. Let $\widetilde{L}_1$ and $\widetilde{L}_2$ be the proper transforms on $\widetilde{S}$ of the lines L_1 and L_2, respectively. Suppose that $Q = \widetilde{L}_1 \cap E_1$ or $\widetilde{L}_2 \cap E_1$. Then*

$$\mathrm{mult}_Q\big(\pi^*(D)\big) \leqslant \frac{49}{27} + \epsilon_k.$$

Proof: Denote by $\widehat{L}_1$, $\widehat{L}_2$, $\widehat{L}_3$ and $\widehat{E}_1$ the proper transforms on $\widehat{S}$ of the curves L_1, L_2, L_3 and E_1, respectively. Then

$$\eta^*(-K_S) \sim_{\mathbb{Q}} \widehat{L}_1 + \widehat{L}_2 + \widehat{L}_3 + 2\widehat{E}_1 + 3E_2.$$

Since the intersection form of the curves $\widehat{L}_1$, $\widehat{L}_2$, $\widehat{L}_3$ and $\widehat{E}_1$ is semi-negative definite, we conclude that $\tau(E_2) = 3$. Then, using Corollary 2.10, we get

$$\mathrm{vol}\big(\eta^*(-K_S) - xE_2\big) = \begin{cases} 3 - \frac{x^2}{2}, & 0 \leqslant x \leqslant 1, \\ \frac{20-4x-x^2}{6}, & 1 \leqslant x \leqslant 2, \\ \frac{12-4x}{3}, & 2 \leqslant x \leqslant 3. \end{cases}$$

Then the required result follows from (4.1). □

Lemma 4.6. *Suppose that $T_P = L + C$, where L is a line, and C is an irreducible conic. Suppose that L and C meet transversally at P. Denote by $\widetilde{L}$ and $\widetilde{C}$ the proper* *transforms on* $\widetilde{S}$ *of the curves* L *and* C*, respectively. Suppose that* $Q = \widetilde{L} \cap E_1$*. Then*

$$\mathrm{mult}_Q\big(\pi^*(D)\big) \leqslant \frac{9}{5} + \epsilon_k.$$

Proof: Denote by $\widehat{L}$, $\widehat{C}$ and $\widehat{E}_1$ the proper transforms on $\widehat{S}$ of the curves L, C and E_1, respectively. Then

$$\eta^*(-K_S) \sim_{\mathbb{Q}} \widehat{L} + \widehat{C} + 2\widehat{E}_1 + 3E_2.$$

Since the intersection form of the curves $\widehat{L}$, $\widehat{C}$ and $\widehat{E}_1$ is negative definite, we conclude that $\tau(E_2) = 3$. Moreover, using Corollary 2.10, we get

$$\mathrm{vol}\big(\eta^*(-K_S) - xE_2\big) = \begin{cases} 3 - \frac{x^2}{2}, & 0 \leqslant x \leqslant 1, \\ \frac{20-4x-x^2}{6}, & 1 \leqslant x \leqslant \frac{14}{5}, \\ 4(3-x)^2, & \frac{14}{5} \leqslant x \leqslant 3. \end{cases}$$

Now the required assertion follows from (4.1). □

Lemma 4.7. *Suppose that* $T_P = L + C$*, where* L *is a line, and* C *is an irreducible conic. Suppose that* L *and* C *meet transversally at* P*. Denote by* $\widetilde{L}$ *and* $\widetilde{C}$ *the proper transforms on* $\widetilde{S}$ *of the curves* L *and* C*, respectively. Suppose that* $Q = \widetilde{C} \cap E_1$*. Then*

$$\mathrm{mult}_Q\big(\pi^*(D)\big) \leqslant \frac{5}{3} + \epsilon_k.$$

Proof: Denote by $\widehat{L}$, $\widehat{C}$ and $\widehat{E}_1$ the proper transforms on $\widehat{S}$ of the curves L, C and E_1, respectively. Then

$$\eta^*(-K_S) \sim_{\mathbb{Q}} \widehat{L} + \widehat{C} + 2\widehat{E}_1 + 3E_2.$$

Since the intersection form of the curves $\widehat{L}$, $\widehat{C}$ and $\widehat{E}_1$ is negative definite, we conclude that $\tau(E_2) = 3$. Moreover, using Corollary 2.10, we get

$$\mathrm{vol}\big(\eta^*(-K_S) - xE_2\big) = \begin{cases} 3 - \frac{x^2}{2}, & 0 \leqslant x \leqslant 2, \\ (3-x)^2 & 2 \leqslant x \leqslant 3. \end{cases}$$

Now the required assertion follows from (4.1). □

Lemma 4.8. *Suppose that* $T_P = L + C$*, where* L *is a line and* C *is an irreducible conic. Suppose that* L *and* C *meet tangentially at* P*. Denote by* $\widetilde{L}$ *and* $\widetilde{C}$ *the proper transforms* *on $\widetilde{S}$ of the curves L and C, respectively. Suppose that $Q = E_1 \cap \widetilde{L} \cap \widetilde{C}$. Then*

$$\mathrm{mult}_Q\big(\pi^*(D)\big) \leqslant \frac{17}{9} + \epsilon_k.$$

Proof: Denote by $\widehat{L}$, $\widehat{C}$ and $\widehat{E}_1$ the proper transforms on $\widehat{S}$ of the curves $\widetilde{L}$, $\widetilde{L}$ and E_1, respectively. Then

$$\eta^*(-K_S) \sim_{\mathbb{Q}} \widehat{L} + \widehat{C} + 2\widehat{E}_1 + 4E_2.$$

Since the intersection form of the curves $\widehat{L}$, $\widehat{C}$ and $\widehat{E}_1$ is negative definite, we conclude that $\tau(E_2) = 4$. Moreover, using Corollary 2.10, we get

$$\mathrm{vol}\big(\eta^*(-K_S) - xE_2\big) = \begin{cases} 3 - \frac{x^2}{2}, & 0 \leqslant x \leqslant 1, \\ \frac{20-4x-x^2}{6}, & 1 \leqslant x \leqslant 2, \\ \frac{(4-x)^2}{3}, & 2 \leqslant x \leqslant 4. \end{cases}$$

Then the required result follows from (4.1). □

Lemma 4.9. *Suppose that T_P is an irreducible cubic. Let $\widetilde{C}$ be the proper transform of the curve C on the surface $\widetilde{S}$. Suppose that $Q \in \widetilde{C}$. Then*

$$\mathrm{mult}_Q\big(\pi^*(D)\big) \leqslant \frac{5}{3} + \epsilon_k.$$

Proof: Denote by $\widehat{C}$ and $\widehat{E}_1$ the proper transforms on $\widehat{S}$ of the curves $\widetilde{C}$ and E_1, respectively. Then

$$\eta^*(-K_S) \sim_{\mathbb{Q}} \widehat{C} + 2\widehat{E}_1 + 3E_2.$$

This gives $\tau(E_2) = 3$, because the intersection form of the curves $\widehat{C}$ and $\widehat{E}_1$ is negative definite. Using Corollary 2.10, we get

$$\mathrm{vol}\big(\eta^*(-K_S) - xE_2\big) = \begin{cases} 3 - \frac{x^2}{2}, & 0 \leqslant x \leqslant 2, \\ (3-x)^2, & 2 \leqslant x \leqslant 3. \end{cases}$$

Then the required result follows from (4.1). □

Now we consider the cases when Q is not contained in the proper transform of the singular curve T_P on the surface $\widetilde{S}$. We start with

Lemma 4.10. *Suppose that* $T_P = L_1 + L_2 + L_3$*, where* L_1*,* L_2 *and* L_3 *are lines passing through* P*. Let* $\widetilde{L}_1$*,* $\widetilde{L}_2$ *and* $\widetilde{L}_3$ *be the proper transforms on* $\widetilde{S}$ *of the lines* L_1*,* L_2 *and* L_3*, respectively. Suppose that* $Q \notin \widetilde{L}_1 \cup \widetilde{L}_2 \cup \widetilde{L}_3$*. Then*

$$\mathrm{mult}_Q\big(\pi^*(D)\big) \leqslant \frac{5}{3} + \epsilon_k.$$

Proof: Denote by $\widehat{L}_1$, $\widehat{L}_2$, $\widehat{L}_3$ and $\widehat{E}_1$ the proper transforms on $\widehat{S}$ of the curves $\widetilde{L}_1$, $\widetilde{L}_2$, $\widetilde{L}_3$ and E_1, respectively. Then

$$\eta^*(-K_S) \sim_{\mathbb{Q}} \widehat{L}_1 + \widehat{L}_2 + \widehat{L}_3 + 3\widehat{E}_1 + 3E_2.$$

This gives $\tau(E_2) = 3$, because the intersection form of the curves $\widehat{L}_1$, $\widehat{L}_2$, $\widehat{L}_3$ and $\widehat{E}_1$ is negative definite. Using Corollary 2.10, we get

$$\mathrm{vol}\big(\eta^*(-K_S) - xE_2\big) = \begin{cases} 3 - \frac{x^2}{2}, & 0 \leqslant x \leqslant 2, \\ (3-x)^2, & 2 \leqslant x \leqslant 3. \end{cases}$$

Then the required result follows from (4.1). □

In the remaining cases, the pseudoeffective threshold $\tau(E_2)$ is not (always) easy to compute. There is a (birational) reason for this. To explain it, observe that the linear system $|-K_{\widetilde{S}}|$ is free from base points and gives a morphism $\phi\colon \widetilde{S} \to \mathbb{P}^2$. Taking its Stein factorization, we obtain a commutative diagram

$$\begin{array}{ccc} \widetilde{S} & \xrightarrow{\ \alpha\ } & \overline{S} \\ {\scriptstyle\pi}\downarrow & \searrow^{\phi} & \downarrow{\scriptstyle\beta} \\ S & \overset{\rho}{\dashrightarrow} & \mathbb{P}^2 \end{array}$$

where α is a birational morphism, β is a double cover branched over a (possibly singular) quartic curve, and ρ is a linear projection from the point P. Here, the surface $\overline{S}$ is a (possibly singular) del Pezzo surface of degree 2. Note that the morphism α is biregular if and only if the curve T_P is irreducible. Moreover, if T_P is reducible, then α-exceptional curves are proper transforms of the lines on S that pass through P.

Let ι be the Galois involution of the double cover β. Then its action lifts to $\widetilde{S}$. On the other hand, this action does not always descent to a (biregular) action of the surface S. Nevertheless, we can always consider ι as a birational involution of the surface S. This involution is known as Geiser involution. It is biregular if and only if P is an Eckardt point of the surface. In this case, the curve E_1 is ι-invariant. However, if P is not an Eckardt point, then $\iota(E_1)$ is the proper transform of the (unique) irreducible component of the curve T_P that is not a line passing through P. In both cases, there exists a commutative diagram

$$\begin{array}{ccc} & \widetilde{S} & \\ {}^{\pi}\swarrow & & \searrow^{\nu} \\ S & \overset{}{\underset{\psi}{\dashrightarrow}} & S' \end{array}$$

where S' is a smooth cubic surface in $\mathbb{P}^3$, which is isomorphic to the surface S via the involution τ, the morphism ν is the contraction of the curve $\iota(E_1)$, and ψ is a birational map given by the linear subsystem in $|-2K_S|$ consisting of all curves having multiplicity at least 3 at the point P.

Let $Q' = \nu(Q)$ and $P' = \nu(\iota(E_1))$. Denote by T'_Q the unique hyperplane section of the cubic surface S' that is singular at Q'. If P is not an Eckardt point and Q is not contained in the proper transform of the curve T_P, then $Q' \neq P'$. In this case, the number $\tau(E_2)$ can be computed using T'_Q. This explains why the remaining cases are (slightly) more complicated.

Lemma 4.11. *Suppose that $T_P = L_1 + L_2 + L_3$, where L_1, L_2 and L_3 are lines such that $P = L_1 \cap L_2$ and $P \notin L_3$. Let $\widetilde{L}_1$, $\widetilde{L}_2$ and $\widetilde{L}_3$ be the proper transforms on $\widetilde{S}$ of the lines L_1, L_2 and L_3, respectively. Suppose that $Q \notin \widetilde{L}_1 \cup \widetilde{L}_2$. Then*

$$\mathrm{mult}_Q\big(\pi^*(D)\big) \leqslant \frac{5}{3} + \epsilon_k.$$

Proof: Denote by $\widehat{L}_1$, $\widehat{L}_2$, $\widehat{L}_3$ and $\widehat{E}_1$ the proper transforms on $\widehat{S}$ of the curves L_1, L_2, L_3 and E_1, respectively. Then

$$\eta^*(-K_S) \sim_{\mathbb{Q}} \widehat{L}_1 + \widehat{L}_2 + \widehat{L}_3 + 2\widehat{E}_1 + 2E_2,$$

which implies that $\tau(E_2) \leqslant 2$. Using Corollary 2.11, we see that

$$\mathrm{vol}\big(\eta^*(-K_S) - xE_2\big) = 3 - \frac{x^2}{2}$$

provided that $0 \leqslant x \leqslant 2$. However, we have $\tau(E_2) > 2$, because the intersection form of the curves $\widehat{L}_1$, $\widehat{L}_2$, $\widehat{L}_3$ and $\widehat{E}_1$ is not semi-negative definite. This also follows from the fact that $\mathrm{vol}(\eta^*(-K_S) - 2E_2) > 0$.

Recall that $\nu\colon \widetilde{S} \to S'$ is the contraction of the curve $\widetilde{L}_3$. We let $L_1' = \nu(\widetilde{L}_1)$, $L_2' = \nu(\widetilde{L}_2)$ and $E_1' = \nu(E_1)$. Then L_1', L_2' and E_1' are coplanar lines on S'.

Since $Q' \in E_1'$, the line E_1' is an irreducible component of the curve T_Q'. Thus, either T_Q' consists of three lines, or T_Q' is a union of the line E_1' and an irreducible conic.

Suppose that $T_Q' = E_1' + Z'$, where Z' is an irreducible conic on S'. Then $Q' \in E_1' \cap Z'$ and $Z' \sim L_1' + L_2'$, which implies that the conic Z' does not meet the lines L_1' and L_2'. Denote by $\widehat{Z}$ the proper transform of the conic Z' on the surface $\widehat{S}$. We have

$$\eta^*(-K_S) \sim_{\mathbb{Q}} \frac{1}{2}\Big(\widehat{Z} + \widehat{L}_1 + \widehat{L}_2\Big) + 2\widehat{E}_1 + \frac{5}{2}E_2.$$

This implies that $\tau(E_2) = \frac{5}{2}$, because the intersection form of the curves $\widehat{Z}$, $\widehat{L}_1$, $\widehat{L}_2$ and $\widehat{E}_1$ is semi-negative definite. Using this $\mathbb{Q}$-rational equivalence and Corollary 2.10, we compute

$$\mathrm{vol}\big(\eta^*(-K_S) - xE_2\big) = \begin{cases} 3 - \frac{x^2}{2}, & 0 \leqslant x \leqslant 2, \\ 5 - 2x, & 2 \leqslant x \leqslant \frac{5}{2}. \end{cases}$$

Thus, a direct computation and (4.1) give

$$\mathrm{mult}_Q\big(\pi^*(D)\big) \leqslant \frac{59}{36} + \epsilon_k < \frac{5}{3} + \epsilon_k,$$

which gives the required assertion.

To complete the proof, we may assume that $T_Q' = E_1' + M' + N'$, where M' and N' are two lines on S' such that $Q' = E_1' \cap M'$. Then $M' + N' \sim L_1' + L_2'$, which implies that the lines M' and N' do not meet the lines L_1' and L_2'. Denote by $\widehat{M}$ and $\widehat{N}$ the proper transforms on the surface $\widehat{S}$ of the lines M' and N', respectively.

Suppose that Q' is also contained in the line N'. This simply means that Q' is an Eckardt point of the surface S'. Then

$$\eta^*(-K_S) \sim_{\mathbb{Q}} \frac{1}{2}\Big(\widehat{M} + \widehat{N} + \widehat{L}_1 + \widehat{L}_2\Big) + 2\widehat{E}_1 + 3E_2.$$

This gives $\tau(E_2) \geqslant 3$. In fact, we have $\tau(E_2) = 3$ here, because the intersection form of the curves $\widehat{M}$, $\widehat{N}$, $\widehat{L}_1$, $\widehat{L}_2$, $\widehat{E}_1$ is negative definite. Using Corollary 2.10, we get

$$\mathrm{vol}\big(\eta^*(-K_S)-xE_2\big)=\begin{cases}3-\frac{x^2}{2}, & 0\leqslant x\leqslant 2,\\ (3-x)^2 & 2\leqslant x\leqslant 3.\end{cases}$$

Now, direct computations and (4.1) give the required inequality.

To complete the proof the lemma, we have to consider the case $Q'\notin N'$. Then

$$\eta^*(-K_S)\sim_{\mathbb{Q}}\frac{1}{2}\Big(\widehat{M}+\widehat{N}+\widehat{L}_1+\widehat{L}_2\Big)+2\widehat{E}_1+\frac{5}{2}E_2.$$

In particular, we see that $\tau(E_2)\geqslant\frac{5}{2}$. Using this $\mathbb{Q}$-rational equivalence and Corollary 2.10, we compute

$$\mathrm{vol}\big(\eta^*(-K_S)-xE_2\big)=\begin{cases}3-\frac{x^2}{2}, & 0\leqslant x\leqslant 2,\\ 7-4x+\frac{x^2}{2}, & 2\leqslant x\leqslant \frac{5}{2}.\end{cases}$$

Thus, in particular, we have $\tau(E_2)>\frac{5}{2}$, since

$$\mathrm{vol}\Big(\eta^*(-K_S)-\frac{5}{2}E_2\Big)=\frac{1}{8}.$$

As in the previous cases, we can find $\tau(E_2)$ and compute $\mathrm{vol}(\eta^*(-K_S)-xE_2)$ for $x>\frac{5}{2}$. However, we can avoid doing this. Namely, note that the divisor $\widehat{E}_1+2\widehat{N}+\widehat{M}$ is nef and

$$\Big(\widehat{E}_1+2\widehat{N}+\widehat{M}\Big)\cdot\Big(\eta^*(-K_S)-xE_2\Big)=6-2x,$$

so that $\tau(E_2)\leqslant 3$. Therefore, using (4.1) and Lemma 2.13, we see that

$$\begin{aligned}\mathrm{mult}_Q\big(\pi^*(D)\big)\leqslant\frac{1}{3}\int_0^{\tau(E_2)}\mathrm{vol}\big(\eta^*(-K_S)-xE_2\big)+\epsilon_k=\\ =\frac{1}{3}\int_0^{\frac{5}{2}}\mathrm{vol}\big(\eta^*(-K_S)-xE_2\big)+\frac{1}{3}\int_{\frac{5}{2}}^{\tau(E_2)}\mathrm{vol}\big(\eta^*(-K_S)-xE_2\big)+\epsilon_k=\\ =\frac{79}{48}+\frac{1}{3}\int_{\frac{5}{2}}^{\tau(E_2)}\mathrm{vol}\big(\eta^*(-K_S)-xE_2\big)+\epsilon_k\leqslant\frac{79}{48}+\frac{\tau(E_2)-\frac{5}{2}}{3}\mathrm{vol}\Big(\eta^*(-K_S)-\frac{5}{2}E_2\Big)+\epsilon_k=\\ =\frac{79}{48}+\frac{\tau(E_2)-\frac{5}{2}}{24}+\epsilon_k\leqslant\frac{79}{48}+\frac{1}{48}+\epsilon_k=\frac{5}{3}+\epsilon_k.\end{aligned}$$

This finish the proof of the lemma. ◻

Lemma 4.12. *Suppose that $T_P = L+C$, where L is a line and C is an irreducible conic. Denote by $\widetilde{L}$ and $\widetilde{C}$ the proper transforms on $\widetilde{S}$ of the curves L and C, respectively. Suppose that $Q \notin \widetilde{L} \cup \widetilde{C}$. Then*

$$\mathrm{mult}_Q(\pi^*(D)) \leqslant \frac{5}{3} + \epsilon_k.$$

Proof: Denote by $\widehat{L}$, $\widehat{C}$ and $\widehat{E}_1$ the proper transforms on $\widehat{S}$ of the curves L, $\widetilde{C}$ and E_1, respectively. Then

$$\eta^*(-K_S) \sim_{\mathbb{Q}} \widehat{L} + \widehat{C} + 2\widehat{E}_1 + 2E_2,$$

so that $\tau(E_2) \geqslant 2$. Using Corollary 2.11, we see that

$$\mathrm{vol}\big(\eta^*(-K_S) - xE_2\big) = 3 - \frac{x^2}{2}$$

provided that $0 \leqslant x \leqslant 2$. Since $\mathrm{vol}(\eta^*(-K_S) - 2E_2) > 0$, we see that $\tau(E_2) > 2$.

Recall that $\nu\colon \widetilde{S} \to S'$ is the contraction of the curve $\widetilde{C}$. Let $L' = \nu(\widetilde{L})$ and $E_1' = \nu(E_1)$. Then L' is a line and E_1' is a conic on S' such that $P' \in L' \cap E_1'$.

First, we suppose that T_Q' is irreducible. Denote by $\widehat{T}_Q$ the proper transform of the cubic T_Q' on the surface $\widehat{S}$. Then $\widehat{T}_Q \cdot \widehat{E}_1 = 0$ and

$$\widehat{T}_Q \cdot \widehat{L} = \widehat{E}_1 \cdot \widehat{L} = 1.$$

Since $\widehat{L}^2 = \widehat{E}_1^2 = -2$ and $\widehat{T}_Q^2 = -1$, we see that the intersection form of the curves $\widehat{L}$, $\widehat{T}_Q$ and $\widehat{E}_1$ is negative definite. On the other hand, we have

$$\eta^*(-K_S) \sim_{\mathbb{Q}} \frac{1}{2}\Big(\widehat{T}_Q + \widehat{L}\Big) + \frac{3}{2}\widehat{E}_1 + \frac{5}{2}E_2.$$

This shows that $\tau(E_2) = \frac{5}{2}$. Hence, using Corollary 2.10, we get

$$\mathrm{vol}\big(\eta^*(-K_S) - xE_2\big) = \begin{cases} 3 - \frac{x^2}{2}, & 0 \leqslant x \leqslant 2, \\ \frac{44-8x-4x^2}{12}, & 2 \leqslant x \leqslant \frac{17}{7}, \\ 4(5-2x)^2, & \frac{17}{7} \leqslant x \leqslant \frac{5}{2}. \end{cases}$$

Then a direct calculation and (4.1) give

$$\mathrm{mult}_Q\big(\pi^*(D)\big) \leqslant \frac{103}{63} + \epsilon_k < \frac{5}{3} + \epsilon_k.$$

Now we suppose that $T'_Q = \ell' + Z'$, where ℓ' is a line, and Z' is an irreducible conic. Denote by $\widehat{\ell}$ and $\widehat{Z}$ the proper transforms on $\widehat{S}$ of the curves ℓ' and Z', respectively. We get

$$\eta^*(-K_S) \sim_{\mathbb{Q}} \frac{1}{2}\Big(\widehat{\ell} + \widehat{Z} + \widehat{L}\Big) + \frac{3}{2}\widehat{E}_1 + \frac{5}{2}E_2.$$

which implies that $\tau(E_2) \geqslant \frac{5}{2}$. Using Corollary 2.10, we get

$$\mathrm{vol}\big(\eta^*(-K_S) - xE_2\big) = \begin{cases} 3 - \frac{x^2}{2}, & 0 \leqslant x \leqslant 2, \\ \frac{34-16x+x^2}{6}, & 2 \leqslant x \leqslant \frac{5}{2}. \end{cases}$$

In particular, we have

$$\mathrm{vol}\Big(\eta^*(-K_S) - \frac{5}{2}E_2\Big) = \frac{1}{24},$$

which implies that $\tau(E_2) > \frac{5}{2}$. Observe that the divisor $\widehat{\ell} + 2\widehat{Z} + \widehat{L}$ is nef and

$$\big(\widehat{\ell} + 2\widehat{Z} + \widehat{L}\big) \cdot \big(\eta^*(-K_S) - xE_2\big) = 9 - 3x,$$

which implies that $\tau(E_2) \leqslant 3$. Thus, using (4.1) and Lemma 2.13, we get

$$\begin{aligned}\mathrm{mult}_Q\big(\pi^*(D)\big) &\leqslant \frac{1}{3}\int_0^{\tau(E_2)} \mathrm{vol}\big(\eta^*(-K_S) - xE_2\big) + \epsilon_k = \\ &= \frac{1}{3}\int_0^{\frac{5}{2}} \mathrm{vol}\big(\eta^*(-K_S) - xE_2\big) + \frac{1}{3}\int_{\frac{5}{2}}^{\tau(E_2)} \mathrm{vol}\big(\eta^*(-K_S) - xE_2\big) + \epsilon_k = \\ &= \frac{709}{432} + \frac{1}{3}\int_{\frac{5}{2}}^{\tau(E_2)} \mathrm{vol}\big(\eta^*(-K_S) - xE_2\big) + \epsilon_k \leqslant \frac{709}{432} + \frac{\tau(E_2) - \frac{5}{2}}{3}\mathrm{vol}\Big(\eta^*(-K_S) - \frac{5}{2}E_2\Big) + \epsilon_k = \\ &= \frac{709}{432} + \frac{\tau(E_2) - \frac{5}{2}}{48} + \epsilon_k \leqslant \frac{709}{432} + \frac{1}{96} + \epsilon_k = \frac{89}{54} + \epsilon_k < \frac{5}{3} + \epsilon_k.\end{aligned}$$

To complete the proof of the lemma, we may assume that $T'_Q = \ell' + M' + N'$, where ℓ', M' and N' are lines such that $Q' \in M' \cap N'$. Since E'_1 is a conic passing through Q', we conclude that Q' is not contained in the line ℓ'. Note that $\ell' \neq L'$, and the lines ℓ', M' and N' do not pass through P'.

Denote by $\widehat{\ell}$, $\widehat{M}$ and $\widehat{N}$ the proper transforms on $\widehat{S}$ of the lines ℓ', M' and N', respectively. We get

$$\eta^*(-K_S) \sim_{\mathbb{Q}} \frac{1}{2}\Big(\widehat{\ell} + \widehat{M} + \widehat{N} + \widehat{L}\Big) + \frac{3}{2}\widehat{E}_1 + \frac{5}{2}E_2,$$

which implies that $\tau(E_2) \geqslant \frac{5}{2}$. In fact, we have $\tau(E_2) > \frac{5}{2}$, because the intersection form of the curves $\widehat{\ell}$, $\widehat{M}$, $\widehat{N}$, $\widehat{L}$ and $\widehat{E}_1$ is not semi-negative definite. Nevertheless, we can use Corollary 2.10 to compute

$$\mathrm{vol}\big(\eta^*(-K_S) - xE_2\big) = \begin{cases} 3 - \frac{x^2}{2}, & 0 \leqslant x \leqslant 2, \\ \frac{92-56x+8x^2}{12}, & 2 \leqslant x \leqslant \frac{5}{2}, \end{cases}$$

so that, in particular, we have

$$\mathrm{vol}\Big(\eta^*(-K_S) - \frac{5}{2}E_2\Big) = \frac{1}{6}.$$

Observe that the divisor $2\widehat{\ell} + \widehat{M} + \widehat{N}$ is nef and

$$\big(2\widehat{\ell} + \widehat{M} + \widehat{N}\big) \cdot \big(\eta^*(-K_S) - xE_2\big) = 6 - 2x,$$

which implies that $\tau(E_2) \leqslant 3$. Thus, using (4.1) and Lemma 2.14, we get

$$\begin{aligned} \mathrm{mult}_Q\big(\pi^*(D)\big) &\leqslant \frac{1}{3}\int_0^{\tau(E_2)} \mathrm{vol}\big(\eta^*(-K_S) - xE_2\big) + \epsilon_k = \\ &= \frac{1}{3}\int_0^{\frac{5}{2}} \mathrm{vol}\big(\eta^*(-K_S) - xE_2\big) + \frac{1}{3}\int_{\frac{5}{2}}^{\tau(E_2)} \mathrm{vol}\big(\eta^*(-K_S) - xE_2\big) + \epsilon_k = \\ &= \frac{89}{54} + \frac{1}{3}\int_{\frac{5}{2}}^{\tau(E_2)} \mathrm{vol}\big(\eta^*(-K_S) - xE_2\big) + \epsilon_k \leqslant \frac{89}{54} + \frac{2}{9}\Big(\tau(E_2) - \frac{5}{2}\Big)\mathrm{vol}\Big(\eta^*(-K_S) - \frac{5}{2}E_2\Big) + \epsilon_k = \\ &= \frac{89}{54} + \frac{2}{54}\Big(\tau(E_2) - \frac{5}{2}\Big) + \epsilon_k \leqslant \frac{89}{54} + \frac{1}{54} + \epsilon_k = \frac{5}{3} + \epsilon_k. \end{aligned}$$

The proof is complete. □

Lemma 4.13. *Suppose that T_P is an irreducible cubic curve. Let $\widetilde{C}$ be its proper transform on the surface $\widetilde{S}$. Suppose that $Q \notin \widetilde{C}$. Then*

$$\mathrm{mult}_Q\big(\pi^*(D)\big) \leqslant \frac{5}{3} + \epsilon_k.$$

Proof: Denote by $\widehat{C}$ and $\widehat{E}_1$ the proper transforms on $\widehat{S}$ of the curves $\widetilde{C}$ and E_1, respectively. Then

$$\eta^*(-K_S) \sim_{\mathbb{Q}} \widehat{C} + 2\widehat{E}_1 + 2E_2.$$

Thus, using Corollary 2.11, we get $\mathrm{vol}(\eta^*(-K_S)-xE_2) = 3-\frac{x^2}{2}$ provided that $0 \leqslant x \leqslant 2$.

Recall that $\nu\colon \widetilde{S} \to S'$ is the contraction of the curve $\widetilde{C}$. Let $E' = \nu(E_1)$. Then E_1' is an irreducible cubic curve that is singular at P'. Thus, the curve E_1' is smooth at the point Q', so that $T_Q' \neq E_1'$. One can easily check that T_Q' does not contain P'.

Suppose that T_Q' is an irreducible cubic. Denote by $\widehat{T}_Q$ the proper transform of the curve T_Q' on the surface $\widehat{S}$. We get $\widehat{E}_1^2 = -2$, $\widehat{T}_Q^2 = -1$, $\widehat{E}_1 \cdot \widehat{T}_Q = 1$ and

$$\eta^*(-K_S) \sim_{\mathbb{Q}} \frac{1}{2}\widehat{T}_Q + \frac{3}{2}\widehat{E}_1 + \frac{5}{2}E_2,$$

which implies that $\tau(E_2) = \frac{5}{2}$. Using Corollary 2.10, we get

$$\mathrm{vol}\big(\eta^*(-K_S) - xE_2\big) = \begin{cases} 3 - \frac{x^2}{2}, & 0 \leqslant x \leqslant \frac{12}{5}, \\ 3(5-2x)^2, & \frac{12}{5} \leqslant x \leqslant \frac{5}{2}. \end{cases}$$

Then (4.1) and direct calculations give

$$\mathrm{mult}_Q\big(\pi^*(D)\big) \leqslant \frac{49}{30} + \epsilon_k < \frac{5}{3} + \epsilon_k.$$

Now we suppose that $T_Q' = \ell' + Z'$, where ℓ' is a line and Z' is an irreducible conic. Denote by $\widehat{\ell}$ and $\widehat{Z}$ the proper transforms on $\widehat{S}$ of the curves ℓ_Q' and Z', respectively. We get

$$\eta^*(-K_S) \sim_{\mathbb{Q}} \frac{1}{2}\Big(\widehat{\ell} + \widehat{Z}\Big) + \frac{3}{2}\widehat{E}_1 + \frac{5}{2}E_2.$$

Since the intersection form of the curves $\widehat{\ell}$, $\widehat{Z}$ and $\widehat{E}_1$ is semi-negative definite, we conclude that $\tau(E_2) = \frac{5}{2}$. Using Corollary 2.10, we get

$$\mathrm{vol}\big(\eta^*(-K_S) - xE_2\big) = \begin{cases} 3 - \frac{x^2}{2}, & 0 \leqslant x \leqslant 2, \\ 5-2x, & 2 \leqslant x \leqslant \frac{5}{2}. \end{cases}$$

Hence, using (4.1), we see that

$$\operatorname{mult}_Q\big(\pi^*(D)\big) \leqslant \frac{59}{36} + \epsilon_k < \frac{5}{3} + \epsilon_k.$$

To complete the proof, we may assume that $T'_Q = \ell' + M' + N'$, where ℓ', M' and N' are lines such that $Q' \in M' \cap N'$. Denote by $\widehat{\ell}$, $\widehat{M}$ and $\widehat{N}$ the proper transforms on $\widehat{S}$ of the lines ℓ', M' and N', respectively. If Q' is contained in the line ℓ', then

$$\eta^*(-K_S) \sim_{\mathbb{Q}} \frac{1}{2}\Big(\widehat{\ell} + \widehat{M} + \widehat{N}\Big) + \frac{3}{2}\widehat{E}_1 + 3E_2,$$

and the intersection form of the curves $\widehat{\ell}$, $\widehat{M}$, $\widehat{N}$ and $\widehat{E}_1$ is negative definite, which implies that $\tau(E_2) = 3$. In this case, Corollary 2.10 gives

$$\operatorname{vol}\big(\eta^*(-K_S) - xE_2\big) = \begin{cases} 3 - \frac{x^2}{2}, & 0 \leqslant x \leqslant 2, \\ (3-x)^2, & 2 \leqslant x \leqslant 3, \end{cases}$$

which implies the required inequality by (4.1).

To complete the proof, we may assume that Q' is not contained in ℓ'. Then the intersection form of the curves $\widehat{\ell}$, $\widehat{M}$, $\widehat{N}$ and $\widehat{E}_1$ is not semi-negative definite. Since

$$\eta^*(-K_S) \sim_{\mathbb{Q}} \frac{1}{2}\Big(\widehat{\ell} + \widehat{M} + \widehat{N}\Big) + \frac{3}{2}\widehat{E}_1 + \frac{5}{2}E_2,$$

we conclude that $\tau(E_2) > \frac{5}{2}$. Moreover, using Corollary 2.10, we get

$$\operatorname{vol}\big(\eta^*(-K_S) - xE_2\big) = \begin{cases} 3 - \frac{x^2}{2}, & 0 \leqslant x \leqslant 2, \\ \frac{x^2-8x+14}{2}, & 2 \leqslant x \leqslant \frac{5}{2}. \end{cases}$$

In particular, we have

$$\operatorname{vol}\Big(\eta^*(-K_S) - \frac{5}{2}E_2\Big) = \frac{1}{8}.$$

Observe that the divisor $2\widehat{\ell} + \widehat{M} + \widehat{N}$ is nef and

$$\big(2\widehat{\ell} + \widehat{M} + \widehat{N}\big) \cdot \big(\eta^*(-K_S) - xE_2\big) = 6 - 2x,$$

which implies that $\tau(E_2)\leqslant 3$. Thus, using (4.1) and Lemma 2.13, we get

$$\mathrm{mult}_Q\big(\pi^*(D)\big)\leqslant\frac{1}{3}\int_0^{\tau(E_2)}\mathrm{vol}\big(\eta^*(-K_S)-xE_2\big)+\epsilon_k=$$
$$=\frac{1}{3}\int_0^{\frac{5}{2}}\mathrm{vol}\big(\eta^*(-K_S)-xE_2\big)+\frac{1}{3}\int_{\frac{5}{2}}^{\tau(E_2)}\mathrm{vol}\big(\eta^*(-K_S)-xE_2\big)+\epsilon_k=$$
$$=\frac{79}{48}+\frac{1}{3}\int_{\frac{5}{2}}^{\tau(E_2)}\mathrm{vol}\big(\eta^*(-K_S)-xE_2\big)+\epsilon_k\leqslant\frac{79}{48}+\frac{\tau(E_2)-\frac{5}{2}}{3}\mathrm{vol}\Big(\eta^*(-K_S)-\frac{5}{2}E_2\Big)+\epsilon_k=$$
$$=\frac{79}{48}+\frac{\tau(E_2)-\frac{5}{2}}{24}+\epsilon_k\leqslant\frac{79}{48}+\frac{1}{48}+\epsilon_k=\frac{5}{3}+\epsilon_k.$$

This completes the proof of the lemma. □

Using Corollary 2.9 and Lemmas 4.4, 4.5, 4.6, 4.7, 4.8, 4.9, 4.10, 4.11, 4.12, 4.13, we immediately get

Corollary 4.14. *We have* $\delta(S)\geqslant\frac{18}{17}$.

4.2 Proof of the main result

In this section, we prove Theorem 4.1. Let S be a smooth cubic surface. We have to prove that $\delta(S)\geqslant\frac{6}{5}$. Fix a positive rational number $\lambda<\frac{6}{5}$. Let D be a k-basis type divisor. To prove Theorem 4.1, it is enough to show that, the log pair $(S,\lambda D)$ is log canonical for $k\gg 1$. Suppose that this is not the case. Then there exists a point $P\in S$ such that $(S,\lambda D)$ is not log canonical at P for $k\gg 1$. Let us seek for a contradiction using results obtained in Section 4.1.

Let $\pi\colon\widetilde{S}\to S$ be the blowup of the point P, and let E_1 be the exceptional divisor of the blow up π. Denote by $\widetilde{D}$ the proper transform of D via π. Then

$$K_{\widetilde{S}}+\lambda\widetilde{D}+\big(\lambda\mathrm{mult}_P(D)-1\big)E_1\sim_{\mathbb{Q}}\pi^*\big(K_S+\lambda D\big).$$

By Corollary 2.8, the log pair $(\widetilde{S},\lambda\widetilde{D}+(\lambda\mathrm{mult}_P(D)-1)E_1)$ is not log canonical at some point $Q\in E_1$. Thus, using Lemma 2.5, we see that

$$\mathrm{mult}_Q\big(\pi^*\big(D\big)\big)=\mathrm{mult}_P\big(D\big)+\mathrm{mult}_Q\big(\widetilde{D}\big)>\frac{2}{\lambda}>\frac{5}{3}.\tag{4.2}$$

Let $\sigma\colon\widehat{S}\to\widetilde{S}$ be the blowup of the point Q, and let E_2 be the exceptional curve of σ.

Denote by $\widehat{D}$ and $\widehat{E}_1$ the proper transforms on $\widehat{S}$ of the divisors $\widetilde{D}$ and E_1, respectively. By Corollary 2.8, the log pair

$$\Big(\widehat{S}, \lambda\widehat{D} + \big(\lambda\mathrm{mult}_P(D) - 1\big)\widehat{E}_1 + \big(\lambda\mathrm{mult}_P(D) + \lambda\mathrm{mult}_Q(\widetilde{D}) - 2\big)E_2\Big)$$

is not log canonical at some point $O \in E_2$.

Let T_P be the hyperplane section of the surface S that is singular at P. Then T_P must be reducible. This follows from (4.2) and Lemmas 4.9 and 4.13.

Denote by $\widetilde{T}_P$ the proper transform of the curve T_P on the surface $\widetilde{S}$. Then $Q \in \widetilde{T}_P$. This follows from (4.2) and Lemmas 4.11 and 4.12.

In the remaining part of this section, we will deal with the following four cases:

1. T_P is a union of three lines passing through P;
2. T_P is a union of three lines and only two of them pass through P;
3. T_P is a union of line and a conic that intersect transversally at P;
4. T_P is a union of line and a conic that intersect tangentially at P.

We will treat each of them in a separate subsection. We start with

4.2.1 Case 1

We have $T_P = L_1 + L_2 + L_3$, where L_1, L_2 and L_3 are lines passing through the point P. We write

$$\lambda D = a_1L_1 + a_2L_2 + a_3L_3 + \Omega,$$

where a_1, a_2 and a_3 are nonnegative rational numbers, and Ω is an effective $\mathbb{Q}$-divisor whose support does not contain L_1, L_2 or L_3. Then

$$L_1 \cdot \Omega = \lambda + a_1 - a_2 - a_3. \tag{4.3}$$

Denote by $\widetilde{L}_1$, $\widetilde{L}_2$ and $\widetilde{L}_3$ the proper transforms on $\widetilde{S}$ of the lines L_1, L_2 and L_3, respectively. We know that $Q \in \widetilde{L}_1 \cup \widetilde{L}_2 \cup \widetilde{L}_3$, so that we may assume that $Q = \widetilde{L}_1 \cap E_1$. Let $\widetilde{\Omega}$ be the proper transform of the divisor Ω on the surface $\widetilde{S}$, and let $m = \mathrm{mult}_P(\Omega)$. Then the log pair

$$\Big(\widetilde{S}, a_1\widetilde{L}_1 + \widetilde{\Omega} + \big(a_1 + a_2 + a_3 + m - 1\big)E_1\Big)$$

is not log canonical at the point Q.

By Lemma 4.3, we have

$$a_1 \leqslant \Big(\frac{5}{9} + \varepsilon_k\Big)\lambda < 1, \tag{4.4}$$

where ε_k is a small constant depending on k such that $\varepsilon_k \to 0$ as $k \to \infty$. Thus, applying Corollary 2.7, we see that

$$L_1 \cdot \Omega + a_1 + a_2 + a_3 - 1 = \widetilde{L}_1 \cdot \Big(\widetilde{\Omega} + \big(a_1 + a_2 + a_3 + m - 1\big)E_1\Big) > 1,$$

which gives $L_1 \cdot \Omega > 2 - a_1 - a_2 - a_3$. Combining this with (4.3), we get

$$a_1 > \frac{2-\lambda}{2}. \tag{4.5}$$

Let $\widetilde{m} = \mathrm{mult}_Q(\widetilde{\Omega})$. Then by Lemma 4.4, we have

$$2a_1 + a_2 + a_3 + m + \widetilde{m} \leqslant \Big(\frac{17}{9} + \epsilon_k\Big)\lambda, \tag{4.6}$$

where ϵ_k is a small constant depending on k such that $\epsilon_k \to 0$ as $k \to \infty$. Then using (4.5) and $m \geqslant \widetilde{m}$, we deduce that

$$\widetilde{m} < \Big(\frac{13}{9} + \frac{\epsilon_k}{2}\Big)\lambda - 1 < 1. \tag{4.7}$$

Denote by $\widehat{L}_1$ and $\widehat{\Omega}$ the proper transforms on $\widehat{S}$ of the divisors $\widetilde{L}_1$ and $\widetilde{\Omega}$, respectively. Then the log pair

$$\Big(\widehat{S}, a_1\widehat{L}_1 + \widehat{\Omega} + \big(a_1 + a_2 + a_3 + m - 1\big)\widehat{E_1} + \big(2a_1 + a_2 + a_3 + m + \widetilde{m} - 2\big)E_2\Big)$$

is not log canonical at the point O.

We claim that $O \in \widehat{L}_1 \cup \widehat{E}_1$. Indeed, we have $(2a_1 + a_2 + a_3 + m + \widetilde{m} - 2) < 1$ by (4.6). Thus, if $O \notin \widehat{L}_1 \cup \widehat{E}_1$, then Corollary 2.7 gives

$$\widetilde{m} = \widehat{\Omega} \cdot E_2 \geqslant \big(\widehat{\Omega} \cdot E_2\big)_O > 1,$$

which is impossible by (4.7). Thus, we have $O \in \widehat{L}_1 \cup \widehat{E}_1$.

If $O \in \widehat{E}_1$, then the log pair

$$\Big(\widehat{S}, \widehat{\Omega} + \big(a_1 + a_2 + a_3 + m - 1\big)\widehat{E_1} + \big(2a_1 + a_2 + a_3 + m + \widetilde{m} - 2\big)E_2\Big)$$

is not log canonical at the point O. Then Corollary 2.7 gives $a_1 + a_2 + a_3 + m + \widetilde{m} > 2$, so that (4.5) and (4.6) gives

$$\Big(\frac{17}{9} + \epsilon_k\Big)\lambda \geqslant 2a_1 + a_2 + a_3 + m + \widetilde{m} > 2 + a_1 > 3 - \frac{\lambda}{2},$$

which is impossible, since $\lambda < \frac{6}{5}$ and $\epsilon_k \to 0$ as $k \to \infty$.

Thus, we see that $O \in \widehat{L}_1$. Then the log pair

$$\Big(\widehat{S}, a_1\widehat{L}_1 + \widehat{\Omega} + \big(2a_1 + a_2 + a_3 + m + \widetilde{m} - 2\big)E_2\Big)$$

is not log canonical at the point O. Now, using (4.6) and (4.7), we have

$$\mathrm{mult}_O\Big(\widehat{\Omega}+\big(2a_1+a_2+a_3+m+\widetilde{m}-2\big)E_2\Big) = 2a_1+a_2+a_3+m+2\widetilde{m}-2 < \Big(\frac{10}{3}+\frac{3\epsilon_k}{2}\Big)\lambda-3 < 1,$$

since $\lambda < \frac{6}{5}$ and $k \gg 1$. Thus, Lemma 2.6 gives

$$L_1 \cdot \Omega + 2a_1 + a_2 + a_3 - 2 = \widehat{L}_1 \cdot \Big(\widehat{\Omega} + \big(2a_1 + a_2 + a_3 + m + \widetilde{m} - 2\big)E_2\Big) > 2 - a_1,$$

so that $L_1 \cdot \Omega + 3a_1 + a_2 + a_3 > 4$. Using (4.3) we get $\lambda + 4a_1 > 4$. Using (4.4), we get

$$\Big(\frac{29}{9} - \varepsilon_k\Big)\lambda > 4,$$

which is impossible, since $\lambda < \frac{6}{5}$ and $\varepsilon_k \to 0$ as $k \to \infty$.

4.2.2 Case 2

We have $T_P = L_1 + L_2 + L_3$, where L_1, L_2 and L_3 are coplanar lines such that $P = L_1 \cap L_2$ and $P \notin L_3$. As in the previous case, we write

$$\lambda D = a_1L_1 + a_2L_2 + \Omega,$$

where a_1 and a_2 are nonnegative rational numbers, and Ω is an effective $\mathbb{Q}$-divisor whose support does not contain the lines L_1 and L_2. Then

$$L_1 \cdot \Omega = \lambda + a_1 - a_2. \tag{4.8}$$

Denote by $\widetilde{L}_1$ and $\widetilde{L}_2$ the proper transforms on $\widetilde{S}$ of the lines L_1 and L_2, respectively. We know that $Q \in \widetilde{L}_1 \cup \widetilde{L}_2$, so that we may assume that $Q = \widetilde{L}_1 \cap E_1$. Let $\widetilde{\Omega}$ be the proper transform of the divisor Ω on the surface $\widetilde{S}$, and let $m = \mathrm{mult}_P(\Omega)$. Then the log pair

$$\Big(\widetilde{S}, a_1\widetilde{L}_1 + \widetilde{\Omega} + \big(a_1 + a_2 + a_3 + m - 1\big)E_1\Big)$$

is not log canonical at the point Q.

By Lemma 4.3, we have

$$a_1 \leqslant \Big(\frac{5}{9} + \varepsilon_k\Big)\lambda < 1, \tag{4.9}$$

where ε_k is a small constant depending on k such that $\varepsilon_k \to 0$ as $k \to \infty$. Thus, using Corollary 2.7, we obtain $L_1 \cdot \Omega > 2 - a_1 - a_2$. Then, using (4.8), we deduce

$$a_1 > \frac{2-\lambda}{2}. \tag{4.10}$$

Let $\widetilde{m} = \mathrm{mult}_Q(\widetilde{\Omega})$. By Lemma 4.5, we have

$$2a_1 + a_2 + m + \widetilde{m} \leqslant \Big(\frac{49}{27} + \epsilon_k\Big)\lambda. \tag{4.11}$$

where ϵ_k is a small constant depending on k such that $\epsilon_k \to 0$ as $k \to \infty$. Thus, using (4.10) and $\widetilde{m} \leqslant m$, we deduce

$$\widetilde{m} < \Big(\frac{38}{27} + \frac{\epsilon_k}{2}\Big)\lambda - 1 < 1. \tag{4.12}$$

Denote by $\widehat{L}_1$ and $\widehat{\Omega}$ the proper transforms on $\widehat{S}$ of the divisors $\widetilde{L}_1$ and $\widetilde{\Omega}$, respectively. Then the log pair

$$\Big(\widehat{S}, a_1\widehat{L}_1 + \widehat{\Omega} + \big(a_1 + a_2 + m - 1\big)\widehat{E}_1 + \big(2a_1 + a_2 + m + \widetilde{m} - 2\big)E_2\Big)$$

is not log canonical at the point O. Then $2a_1 + a_2 + m + \widetilde{m} - 2 < 1$ by (4.11). Thus, using (4.12) and arguing as in Subsection 4.2.1, we see that $O \in \widehat{L}_1 \cup \widehat{E}_1$.

If $O \in \widehat{E}_1$, then the log pair

$$\Big(\widehat{S}, \widehat{\Omega} + \big(a_1 + a_2 + m - 1\big)\widehat{E}_1 + \big(2a_1 + a_2 + m + \widetilde{m} - 2\big)E_2\Big)$$

is not log canonical at the point O, so that $a_1 + a_2 + m + \widetilde{m} > 2$ by Corollary 2.7. Hence, using (4.10) and (4.11), we get

$$\Big(\frac{49}{27} + \epsilon_k\Big)\lambda \geqslant 2a_1 + a_2 + m + \widetilde{m} > 2 + a_1 > 3 - \frac{\lambda}{2},$$

which is impossible, since $\lambda < \frac{6}{5}$ and $\epsilon_k \to 0$ as $k \to \infty$.

We see that $O \in \widehat{L}_1$. Then the log pair

$$\Big(\widehat{S}, a_1\widehat{L}_1 + \widehat{\Omega} + \big(2a_1 + a_2 + m + \widetilde{m} - 2\big)E_2\Big)$$

is not log canonical at the point O. Now, using (4.11) and (4.12), we deduce

$$\mathrm{mult}_O\Big(\widehat{\Omega} + \big(2a_1 + a_2 + m + \widetilde{m} - 2\big)E_2\Big) = 2a_1 + a_2 + m + 2\widetilde{m} - 2 < \Big(\frac{29}{9} + \frac{3\epsilon_k}{2}\Big)\lambda - 3 < 1,$$

because $\lambda < \frac{6}{5}$ and $k \gg 1$. Then we may apply Lemma 2.6 to get

$$L_1 \cdot \Omega + 2a_1 + a_2 - 2 = \widehat{L}_1 \cdot \Big(\widehat{\Omega} + \big(2a_1 + a_2 + m + \widetilde{m} - 2\big)E_2\Big) > 2 - a_1,$$

so that $L_1 \cdot \Omega + 3a_1 + a_2 > 4$. Using (4.8) we get $\lambda + 4a_1 > 4$. Then, by (4.9), we have

$$\Big(\frac{29}{9} - \varepsilon_k\Big)\lambda > 4,$$

which is impossible, since $\lambda < \frac{6}{5}$ and $\varepsilon_k \to 0$ as $k \to \infty$.

4.2.3 Case 3

We have $T_P = L + C$, where L is a line and C is an irreducible conic such that they intersect transversally at P. As in the previous cases, we write

$$\lambda D = aL + bC + \Omega,$$

where a and b are nonnegative rational numbers, and Ω is an effective $\mathbb{Q}$-divisor whose support does not contain the curves L and C. Then Lemma 4.3 gives us

$$a_1 \leqslant \Big(\frac{5}{9}+\varepsilon_k\Big)\lambda < 1, \tag{4.13}$$

where ε_k is a small constant depending on k such that $\varepsilon_k \to 0$ as $k \to \infty$. And also, we have

$$L \cdot \Omega = \lambda + a - 2b. \tag{4.14}$$

Denote by $\widetilde{L}$ and $\widetilde{C}$ the proper transforms on $\widetilde{S}$ of the curves L and C, respectively. We know that $Q \in \widetilde{L} \cup \widetilde{C}$. Moreover, using (4.2) and Lemma 4.7, we see that $Q = \widetilde{L} \cap E_1$.

Denote by $\widetilde{\Omega}$ the proper transforms on $\widetilde{S}$ of the divisor Ω. Let $m = \mathrm{mult}_P(\Omega)$. Then the log pair

$$\Big(\widetilde{S}, a\widetilde{L} + \widetilde{\Omega} + \big(a+b+m-1\big)E_1\Big)$$

is not log canonical at Q. Since $a < 1$, we can apply Corollary 2.7 to this log pair and the curve $\widetilde{L}$. This gives $L \cdot \Omega > 2-a-b$. Combining this with (4.14), we have $\lambda + 2a - b > 2$, so that

$$a > \frac{2+b-\lambda}{2} \geqslant \frac{2-\lambda}{2}. \tag{4.15}$$

Let $\widetilde{m} = \mathrm{mult}_Q(\widetilde{\Omega})$. Then Lemma 4.6 gives

$$2a + b + m + \widetilde{m} \leqslant \Big(\frac{9}{5} + \epsilon_k\Big)\lambda, \tag{4.16}$$

where ϵ_k is a small constant depending on k such that $\epsilon_k \to 0$ as $k \to \infty$. Thus, using (4.15) and $\widetilde{m} \leqslant m$, we deduce that

$$\widetilde{m} < \Big(\frac{7}{5} + \frac{\epsilon_k}{2}\Big)\lambda - 1 < 1. \tag{4.17}$$

Denote by $\widehat{L}$ and $\widehat{\Omega}$ the proper transforms on $\widehat{S}$ of the divisors $\widetilde{L}$ and $\widetilde{\Omega}$, respectively. Then the log pair

$$\Big(\widehat{S}, a\widehat{L} + \widehat{\Omega} + \big(a+b+m-1\big)\widehat{E}_1 + \big(2a+b+m+\widetilde{m}-2\big)E_2\Big)$$

is not log canonical at the point O. Note that $2a + b + m + \widetilde{m} - 2 < 1$ by (4.16). Thus, using (4.17) and arguing as in Subsection 4.2.1, we see that $O \in \widehat{L} \cup \widehat{E_1}$.

If $O \in \widehat{E_1}$, then the log pair

$$\Big(\widehat{S}, \widehat{\Omega} + \big(a+b+m-1\big)\widehat{E}_1 + \big(2a+b+m+\widetilde{m}-2\big)E_2\Big)$$

is not log canonical at O. Applying Corollary 2.7 again, we obtain $a+b+m+\widetilde{m} > 2$, so that (4.15) and (4.16) give

$$\Big(\frac{9}{5}+\epsilon_k\Big)\lambda \geqslant 2a+b+m+\widetilde{m} > 2+a > 3-\frac{\lambda}{2},$$

which is impossible, since $\lambda < \frac{6}{5}$ and $\epsilon_k \to 0$ as $k \to \infty$.

We see that $O \in \widehat{L}$. Then the log pair

$$\Big(\widehat{S}, a\widehat{L} + \widehat{\Omega} + \big(2a+b+m+\widetilde{m}-2\big)E_2\Big)$$

is not log canonical at the point O. Now using (4.16) and (4.17), we obtain

$$\mathrm{mult}_O\Big(\widehat{\Omega} + \big(2a+b+m+\widetilde{m}-2\big)E_2\Big) = 2a+b+m+2\widetilde{m}-2 < \Big(\frac{12}{5}+\frac{3\epsilon_k}{2}\Big)\lambda - 3 < 1,$$

because $\lambda < \frac{6}{5}$ and $\epsilon_k \to 0$ as $k \to \infty$. Thus, applying Lemma 2.6, we get

$$L\cdot\Omega + 2a+b-1 = \widehat{L}\cdot\Big(\widehat{\Omega} + \big(2a+b+m+\widetilde{m}-2\big)E_2\Big) > 2-a,$$

which gives $L\cdot\Omega + 3a + b > 4$. Using (4.14), we get $\lambda + 4a > 4 + b \geqslant 4$, so that (4.13) implies that

$$\Big(\frac{29}{9}-\varepsilon_k\Big)\lambda > 4,$$

which is impossible, since $\lambda < \frac{6}{5}$ and $\varepsilon_k \to 0$ as $k \to \infty$.

4.2.4 Case 4

We have $T_P = L + C$, where L is a line, and C is an irreducible conic that tangents L at the point P. We write

$$\lambda D = aL + bC + \Omega,$$

where a and b are nonnegative rational numbers, and Ω is an effective $\mathbb{Q}$-divisor whose support does not contain L and C. Let $m = \mathrm{mult}_P(\Omega)$. Then

$$a + b + m > 1 \tag{4.18}$$

by Lemma 2.5. Meanwhile, it follows from Lemma 4.3 that

$$a_1 \leqslant \Big(\frac{5}{9} + \varepsilon_k\Big)\lambda < 1, \tag{4.19}$$

where ε_k is a small constant depending on k such that $\varepsilon_k \to 0$ as $k \to \infty$. And also, we have

$$L \cdot \Omega = \lambda + a - 2b. \tag{4.20}$$

Denote by $\widetilde{L}$ and $\widetilde{C}$ the proper transforms on $\widetilde{S}$ of the curves L and C, respectively. We know that $Q = \widetilde{L} \cap \widetilde{C}$. Denote by $\widetilde{\Omega}$ the proper transforms on $\widetilde{S}$ of the divisor Ω. Then the log pair

$$\Big(\widetilde{S}, a\widetilde{L} + b\widetilde{C} + \widetilde{\Omega} + \big(a + b + m - 1\big)E_1\Big)$$

is not log canonical at the point Q. Since $a < 1$ by (4.19), we may apply Corollary 2.7 to this log pair at Q with respect to the curve $\widetilde{L}$. This gives

$$L \cdot \Omega > 2 - a - 2b.$$

Combining this with (4.20), we get $\lambda + 2a > 2$, so that

$$a > \frac{2 - \lambda}{2}. \tag{4.21}$$

Let $\widetilde{m} = \mathrm{mult}_Q(\widetilde{\Omega})$. Then Lemma 4.8 gives

$$2a + 2b + m + \widetilde{m} = \lambda \cdot \mathrm{mult}_Q(\pi^*(D)) \leqslant \Big(\frac{17}{9} + \epsilon_k\Big)\lambda. \tag{4.22}$$

where ϵ_k is a small constant depending on k such that $\epsilon_k \to 0$ as $k \to \infty$. Thus, using (4.21) and $\widetilde{m} \leqslant m$, we deduce that

$$\widetilde{m} < \Big(\frac{13}{9} + \frac{\epsilon_k}{2}\Big)\lambda - 1 < 1. \tag{4.23}$$

Denote by $\widehat{L}$, $\widehat{C}$ and $\widehat{\Omega}$ the proper transforms on $\widehat{S}$ of the divisors $\widetilde{L}$, $\widetilde{C}$ and $\widetilde{\Omega}$, respectively. Then the log pair

$$\Big(\widehat{S}, a\widehat{L}+b\widehat{C}+\widehat{\Omega}+\big(a+b+m-1\big)\widehat{E}_1+\big(2a+2b+m+\widetilde{m}-2\big)E_2\Big)$$

is not log canonical at O. Moreover, it follows from (4.22) that $2a+2b+m+\widetilde{m}-2<1$. Thus, using (4.23) and arguing as in Subsection 4.2.1, we see that $O\in\widehat{L}\cup\widehat{C}\cup\widehat{E_1}$.

If $O\in\widehat{E_1}$, then the log pair

$$\Big(\widehat{S}, \widehat{\Omega}+\big(a+b+m-1\big)\widehat{E}_1+\big(2a+2b+m+\widetilde{m}-2\big)E_2\Big)$$

is not log canonical at O. In this case, Corollary 2.7 applied to this log pair (and the curve E_2) gives $a+b+m+\widetilde{m}>2$, so that (4.21) and (4.16) give

$$\Big(\frac{17}{9}+\epsilon_k\Big)\lambda\geqslant 2a+2b+m+\widetilde{m}>2+a+b>3-\frac{\lambda}{2},$$

which is impossible, since $\lambda<\frac{6}{5}$ and $\epsilon_k\to 0$ as $k\to\infty$.

If $O\in\widehat{C}$, then the log pair

$$\Big(\widehat{S}, b\widehat{C}+\widehat{\Omega}+\big(2a+2b+m+\widetilde{m}-2\big)E_2\Big)$$

is not log canonical at O. In this case, if we apply Corollary 2.7 to this log pair with respect to E_2, we get $b+\widetilde{m}>1$, so that (4.22) gives

$$2a+b+m+1<\Big(\frac{17}{9}+\epsilon_k\Big)\lambda-1.$$

Combining this with (4.18)), we see that $a<(\frac{17}{9}+\epsilon_k)\lambda-2$, so that (4.21) gives

$$\Big(\frac{43}{18}+\epsilon_k\Big)\lambda>3,$$

which is impossible, since $\lambda<\frac{6}{5}$ and $\epsilon_k\to 0$ as $k\to\infty$.

We see that $O\in\widehat{L}$. Then the log pair

$$\Big(\widehat{S}, a\widehat{L}+\widehat{\Omega}+\big(2a+2b+m+\widetilde{m}-2\big)E_2\Big)$$

is not log canonical at the point O. Now using (4.22), (4.23) and $\lambda < \frac{6}{5}$, we deduce that

$$\mathrm{mult}_O\Big(\widehat{\Omega} + \big(2a+2b+m+\widetilde{m}-2\big)E_2\Big) = 2a+2b+m+2\widetilde{m}-2 < \Big(\frac{10}{3}+\frac{3\epsilon_k}{2}\Big)\lambda - 3 < 1.$$

since $\lambda < \frac{6}{5}$ and $k \to \infty$. Then we may apply Lemma 2.6 to get

$$L\cdot\Omega + 2a + 2b - 2 = \widehat{L}\cdot\Big(\widehat{\Omega} + \big(2a+2b+m+\widetilde{m}-2\big)E_2\Big) > 2-a,$$

which gives $L\cdot\Omega + 3a + 2b > 4$. Using (4.20), we see that $\lambda + 4a > 4$, so that (4.19) gives

$$\Big(\frac{29}{9} - \varepsilon_k\Big)\lambda > 4,$$

which is impossible, since $\lambda < \frac{6}{5}$ and $\varepsilon_k \to 0$ as $k \to \infty$.

The proof of Theorem 4.1 is complete.

Chapter 5 Delta invariants on asymptotically log Fano surfaces

In this chapter we study δ-invaraints on log Fano surfaces. We will be mainly interested in a special family of asymptotically log del Pezzo surfaces, which is conjectured to admit Kähler-Einstein edge metrics (cf. [14]). We partially verify this conjecture from the algebraic side, by showing that, a general element of this family is uniformly log K-stable.

5.1 Basic set-up

Let $\overline{S} = \mathbb{P}^1 \times \mathbb{P}^1$, and let $\overline{C}$ be a smooth curve of bi-degree $(1, 2)$ in $\overline{S}$. Then $\overline{S}$ contains exactly two curves of bi-degree $(1, 0)$ that are tangent to $\overline{C}$. Denote them by $\overline{F}_0$ and $\overline{F}_\infty$. Then each intersection $\overline{F}_0 \cap \overline{C}$ and $\overline{F}_\infty \cap \overline{C}$ consists of one point. Let $\overline{F}_1, \ldots, \overline{F}_r$ be

$$r \geq 7$$

distinct curves in $\overline{S}$ of bi-degree $(1, 0)$ that are all different from the curves $\overline{F}_0$ and $\overline{F}_\infty$. Then each intersection $\overline{F}_i \cap \overline{C}$ consists of two points. Let $\overline{P}_i$ be one of these two points.

Let $\pi\colon S \to \overline{S}$ be blow up of the points $\overline{P}_1, \ldots, \overline{P}_r$, and let C be the proper transform of the curve $\overline{C}$. Then

$$C^2 = \overline{C}^2 - r = 4 - r < 0,$$

since we assume that $r \geq 7$. This shows that the curve C is contained in the boundary of the Mori cone of the surface S. Moreover, it is not hard to check that F_0, $C + F_1$ and $C + E_1$ are in the boundary of the Mori cone as well.

Denote by E_i the exceptional curve of the blow up π such that $\pi(E_i) = \overline{P}_i$. Similarly, denote by $F_0, F_1, \ldots, F_r, F_\infty$ the proper transform on the surface S of the curves $\overline{F}_0, \overline{F}_1, \ldots, \overline{F}_r, \overline{F}_\infty$. Also denote by F a general curve in the pencil $|F_0|$. Finally, let

$$L = -\Big(K_S + (1 - \beta)C\Big),$$

where β is a sufficiently small positive rational number. Then

$$L \sim_{\mathbb{Q}} F + \beta C,$$

which implies that L is ample for $\beta < \frac{2}{r-4}$. So in particular, $(S, (1-\beta)C)$ is a log Fano pair for sufficiently small β.

The main goal of this chapter is to show the following

Theorem 5.1. *The log pair $(S, (1-\beta)C)$ is uniformly log K-stable for sufficiently small cone angle β.*

By the recent work of Tian-Wang [59] and Berman-Blum-Jonsson [7], this result implies that the log pair $(S, (1-\beta)C)$ admits Kähler-Einstein edge metrics with cone angle β along C when β is small enough. We will prove Theorem 5.1 by showing that $\delta(S, (1-\beta)C) > 1$ (cf. Theorem 1.3,5.6). To this end, we first need several multiply estimates.

5.2 Multiplicity estimates

We use the same notation as in the previous section. Suppose that

$$D \sim_{\mathbb{Q}} L$$

is any n-basis type divisor of L. with $n \gg 1$. Let Z be a smooth curve on the surface S. We will write

$$D = aZ + \Delta,$$

where $a \geq 0$, Δ is an effective divisor and Z is not contained in the support of Δ. Our goal is to estimate a from above. By Theorem 2.12, we know that

$$a \leq \frac{1}{L^2} \int_0^{\tau(Z)} \mathrm{vol}(L - xZ) dx + \epsilon_n,$$

where $\epsilon_n \to 0$ as $n \to \infty$.

Lemma 5.2. *Let Z be an irreducible curve in $|F|$. Then*

$$a \leq \frac{1}{2} - \frac{\beta(r-4)}{8} + \frac{5(r-4)^2\beta^2}{96} + O(\beta^3) + \epsilon_n.$$

Proof: Since $L \sim_{\mathbb{Q}} F + \beta C$ and $C^2 < 0$, we have

$$\tau(L, Z) = 1$$

To find $\sigma(L, Z)$, we compute

$$\Big(L - \lambda Z\Big) \cdot C = (1-\lambda) F \cdot C + \beta C^2 = 2(1-\lambda) - \beta(r-4),$$

where $\lambda \in \mathbb{Q}$. This gives

$$\sigma(L, Z) = 1 - \beta \frac{r-4}{2} \leq 1 = \tau(L, Z).$$

Then the result follows from Theorem 2.12. Indeed, we have

$$\begin{aligned} a &\leq \frac{1}{L^2} \int_0^1 \mathrm{vol}(L - xZ) dx + \epsilon_n \\ &= \frac{1}{L^2} \int_0^{1 - \frac{r-4}{2}\beta} (L - xZ)^2 dx + \frac{\beta^3}{3}(r-4)^2 + \epsilon_n \\ &= \frac{2\beta - \beta^2(r-4) + \frac{\beta^3 (r-4)^2}{3}}{4\beta - \beta^2(r-4)} + \epsilon_n \\ &= \frac{1}{2} - \frac{(r-4)\beta}{8} + \frac{5(r-4)^2\beta^2}{96} + O(\beta^3) + \epsilon_n \end{aligned}$$

□

Lemma 5.3. *Let Z be one of the π-exceptional curves. Then*

$$a \leq \frac{1}{2} - \frac{\beta(r-6)}{8} + O(\beta^2) + \epsilon_n$$

for some constant ϵ that depends only on the classed of L and Z in $\mathrm{Pic}(S)$.

Proof: Without loss of generality, we may assume that $Z = E_1$. Since $L \sim_{\mathbb{Q}} E_1 + F_1 + \beta C$ and $F_1 + \beta C$ is on the boundary of the Mori cone, we have $\tau(L, Z) = 1$. To find $\sigma(L, Z)$, observe that $F \sim E_1 + F_1$, so that

$$L - \lambda Z \sim_{\mathbb{Q}} (1-\lambda) Z + F_1 + \beta C,$$

where $\lambda \in \mathbb{Q}$. Thus, we compute

$$\Big(L - \lambda Z\Big) \cdot F_1 = \beta - \lambda.$$

Similarly, we have

$$\Big(L - \lambda Z\Big) \cdot C = (1-\lambda)F \cdot C + \beta C^2 = 2 - \beta(r-4) - \lambda,$$

This gives $\sigma(L, Z) = \beta$, since β is assumed to be sufficiently small.

Now, let us use Theorem 2.12. It gives

$$a \leq \frac{1}{L^2}\int_0^1 \mathrm{vol}(L - xZ)dx + \epsilon_n.$$

We set

$$\mu = 1 - \frac{\beta(r-5)}{2}.$$

We aligned the estimate in three pieces:

$$a \leq \frac{1}{L^2}(\int_0^\beta \mathrm{vol}(L - xZ)dx + \int_\beta^\mu \mathrm{vol}(L - xZ)dx + \int_\mu^1 \mathrm{vol}(L - xZ)dx) + \epsilon_n$$

- For the first piece, since $L - xZ$ is nef for $x \in [0, \beta]$, we have

$$\int_0^\beta \mathrm{vol}(L - xZ)dx = \int_0^\beta (L - xZ)^2 dx$$

- For the second piece, we use Lemma 2.10. Notice that

$$(L - xZ) \cdot F_1 = \beta - x.$$

  So we get

$$\mathrm{vol}(L - xZ) = \mathrm{vol}(L - xZ - (x-\beta)F_1),\ x \geq \beta.$$

  In other words, to calculate the volume, we can replace the line bundle $L - xZ$ by $L - xZ - (x-\beta)F_1$ when $x \geq \beta$. Now, observe that

$$L - xZ - (x-\beta)F_1 \sim (1-x)F_0 + \beta C + \beta F_1.$$

So $L - xZ - (x-\beta)F_1$ is nef if and only if

$$((1-x)F_0 + \beta C + \beta F_1) \cdot C \geq 0.$$

So we see that, $L - xZ - (x-\beta)F_1$ is nef if and only if

$$x \leq 1 - \frac{\beta}{2}(r-5) = \mu.$$

Therefore, for $\beta \leq x \leq \mu$, we get

$$\mathrm{vol}(L - xZ) = (L - xZ - (x-\beta)F_1)^2 = 4\beta(1-x) - \beta^2(r-5).$$

So we have

$$\int_\beta^\mu \mathrm{vol}(L - xZ) = \int_\beta^\mu (4\beta(1-x) - \beta^2(r-5))dx.$$

- For the third piece, we clearly have

$$\int_\mu^1 \mathrm{vol}(L - xZ)dx \leq (1-\mu)\mathrm{vol}(L - \mu Z) = \frac{\beta^3(r-5)^2}{2}$$

Summing up the three pieces, we get

$$a \leq \frac{2-(r-5)\beta + O(\beta^2)}{4-(r-4)\beta} + \epsilon_n = \frac{1}{2} + \frac{6-r}{8}\beta + O(\beta^2) + \epsilon_n.$$

□

Lemma 5.4. *Let Z be one of the curves $F_1, \ldots, F_r$. Then*

$$a \leq \frac{1}{2} + \frac{6-r}{8}\beta + O(\beta^2) + \epsilon_n.$$

Proof: The proof is exactly the same as above since E_i and F_i are symmetric in our calculations. □

Lemma 5.5. *Let Z be the curve C. Then*

$$a \leq \frac{\beta}{2} + \frac{r-4}{24}\beta^2 + O(\beta^3) + \epsilon_n.$$

Proof: We have

$$\sigma(L,Z)=\tau(L,Z)=\beta$$

in this case, so by Theorem 2.12, we have

$$a\le \frac{1}{L^2}\int_0^{\beta}(L-xZ)^2dx+\epsilon_n.$$

Simplifying, we get (this holds for any $r\ge 1$)

$$a\le \frac{\beta}{2}+\frac{r-4}{24}\beta^2+O(\beta^3)+\epsilon_n.$$

□

5.3 Proof of the main result

We fix $\beta>0$, which is a sufficiently small rational number. We also fix $r\ge 7$. We use the same notation as in the Section 5.1. The main result of this section is the following.

Theorem 5.6. *One has*

$$\delta(S,(1-\beta)C)>1.$$

The proof of this theorem uses standard techniques from [11]. We fix a constant $\lambda>1$ which is sufficiently close to 1, say

$$\lambda=1+\frac{\beta}{100}.$$

In this section, D will always denote a k-basis type divisor of the $\mathbb{Q}$-line bundle $-K_S-(1-\beta)C$. Here we assume that k is sufficiently large.

To prove Theorem 5.6, it is enough to show that

$$(S,(1-\beta)C+\lambda D)$$

is log canonical for any k-basis type divisor D with $k\gg 1$. We argue by contradiction.

Suppose that there exists a k-basis type divisor D with $k\gg 1$ such that the log pair

$$(S,(1-\beta)C+\lambda D)$$

is not lc at some point $P \in S$. We will seek for a contradiction. We split the argument into several lemmas.

Lemma 5.7. *The point p is contained in $C \cup E_1 \cup \cdots \cup E_r \cup F_1 \cup \cdots \cup F_r$.*

Proof: If this is not the case, then let $Z \in |F_0|$ be the curve that contains p. Then Z is irreducible and smooth. We write

$$\lambda D = aZ + \Delta.$$

Then by inversion of adjunction, we have

$$2\lambda\beta = Z \cdot \Delta > 1,$$

which is a contradiction since $\lambda\beta$ is small. □

Lemma 5.8. *The point p is contained in the curve C.*

Proof: If this is not the case, then suppose that $P \in E_1$. We write

$$\lambda D = aE_1 + \Delta.$$

By Lemma 5.3 we may assume that

$$a \leq \frac{2}{3}.$$

On the other hand, by inversion of adjunction at the point p, we have

$$a + \lambda\beta = E_1 \cdot \Delta > 1,$$

which is a contradiction since β is assumed to be sufficiently small. The same argument works for any other E_i and F_i. □

Lemma 5.9. *The point p is contained in $F_0 \cup F_1 \cup \cdots \cup F_r \cup E_1 \cup \cdots \cup E_r$.*

Proof: If this is not the case, then $P \in C$ is a general point. Let $Z \in |F_0|$ be the curve that contains p. Then Z intersects C transversely at p. We write

$$\lambda D = aZ + \epsilon C + \Omega.$$

By our assumption, the log pair

$$(S, (1-\beta+\epsilon)C + aZ + \Omega)$$

is not lc at p. By Lemma 5.2 and Lemma 5.5, we may assume that

$$0 \le a \le \frac{2}{3},\ 0 \le \epsilon \le \frac{2\beta}{3}.$$

If we put

$$m = \mathrm{mult}_p\Omega,$$

then we have

$$m \le (Z \cdot \Omega)_p \le 2(\lambda\beta - \epsilon).$$

It is also clear that

$$(C \cdot \Omega)_p \le C \cdot \Omega = 2\lambda - 2a - (\lambda\beta - \epsilon)(r-4).$$

Now we apply Theorem 2.16 at the point p. We get

$$(C \cdot \Omega)_p > \frac{(Z \cdot \Omega)_p}{(Z \cdot \Omega)_p - (\beta - \epsilon)}(1-a) - (\beta - \epsilon).$$

Thus we have

$$2\lambda - 2a - (\lambda\beta - \epsilon)(r-4) > \frac{2(\lambda\beta - \epsilon)}{2(\lambda\beta - \epsilon) - (\beta - \epsilon)}(1-a) - (\beta - \epsilon).$$

Rearranging this, we get

$$(2-\beta)(\lambda - 1) + \frac{2(\lambda-1)(1-a)\beta}{(2\lambda - 1)\beta - \epsilon} > (\lambda\beta - \epsilon)(r-5).$$

Using $\epsilon \le \frac{2\beta}{3}$, we easily deduce that

$$(8 - \beta - 6a)(\lambda - 1) > (\lambda - \frac{2}{3})(r-5)\beta.$$

which is impossible for $r \ge 6$ and $\lambda = 1 + \frac{\beta}{100}$.

□

Lemma 5.10. *The point p is contained in $F_0 \cup F_\infty$.*

Proof: Suppose that this is not the case. Then without loss of generality, we may assume that p is contained in $E_1 \cup F_1$. We decompose λD as

$$\lambda D = a_E E_1 + a_F F_1 + \epsilon C + \Omega.$$

We may assume that $P = E_1 \cap C$ (the proof for $P = F_1 \cap C$ is exactly the same). Then by our assumption, the log pair

$$(S, (1-\beta+\epsilon)C + a_E E_1 + \Omega)$$

is not lc at the point p. By Lemma 5.3 and Lemma 5.5, we may assume that

$$0 \leq a_E \leq \frac{2}{3},\ 0 \leq \epsilon \leq \frac{2\beta}{3}.$$

We set

$$m = \mathrm{mult}_p \Omega.$$

Notice that we have

$$\lambda\beta - \epsilon + a_E - a_F = E_1 \cdot \Omega \geq m,$$

$$\lambda\beta - \epsilon - a_E + a_F = F_1 \cdot \Omega \geq 0.$$

From these two inequalities we get

$$m \leq 2(\lambda\beta - \epsilon),\ a_E - a_F \leq \lambda\beta - \epsilon.$$

In the meantime, it is also clear that

$$(C \cdot \Omega)_p \leq C \cdot \Omega = 2\lambda - a_E - a_E - (\lambda\beta - \epsilon)(r-4),$$

$$(E_1 \cdot \Omega)_p \leq E_1 \cdot \Omega = \lambda\beta - \epsilon + a_E - a_F \leq 2(\lambda\beta - \epsilon).$$

Now we apply Theorem 2.16 at the point p. We get

$$(C \cdot \Omega)_p > \frac{(E_1 \cdot \Omega)_p}{(E_1 \cdot \Omega)_p - (\beta - \epsilon)}(1 - a_E) - (\beta - \epsilon).$$

Then we have

$$2\lambda - a_E - a_F - (\lambda\beta - \epsilon)(r-4) > \frac{2(\lambda\beta - \epsilon)}{2(\lambda\beta - \epsilon) - (\beta - \epsilon)}(1 - a_E) - (\beta - \epsilon).$$

Rearranging this, we get

$$(2-\beta)(\lambda - 1) + \frac{2(\lambda - 1)(1 - a_E)\beta}{(2\lambda - 1)\beta - \epsilon} > (\lambda\beta - \epsilon)(r-5) - (a_E - a_F).$$

Using $a_E - a_F \leq (\lambda\beta - \epsilon)$ and $\epsilon \leq \frac{2\beta}{3}$, we easily see that

$$(8 - \beta - 6a_E)(\lambda - 1) > (\lambda - \frac{2}{3})(r-6)\beta,$$

which is impossible for $r \geq 7$ and $\lambda = 1 + \frac{\beta}{100}$.

$\square$

With all the above lemmas combined, we may assume that

$$P = F_0 \cap C.$$

We write

$$\lambda D = aF_0 + \epsilon C + \Omega.$$

To get a contradiction for this case, here we only require $r \geq 5$. By Lemma 5.2 and Lemma 5.5, we may assume that

$$0 \leq 2a \leq 1 - \frac{\beta}{5},\ 0 \leq \epsilon \leq \frac{2\beta}{3}.$$

Note that, here we used the fact that β and ϵ_k are sufficiently small. We set

$$m = \mathrm{mult}_p \Omega.$$

By our assumption, the log pair

$$(S, (1 - \beta + \epsilon)C + aF_0 + \Omega)$$

is not lc at p.

We let $g: \tilde{S} \to S$ be the blow-up of the point p, and let G be the exceptional curve of g. We let $\tilde{C}$, $\tilde{F_0}$ and $\tilde{\Omega}$ be the proper transform of C, F_0 and Ω respectively on the surface $\tilde{S}$. Let us put

$$\tilde{P} = \tilde{C} \cap G,\ \tilde{m} = \mathrm{mult}_{\tilde{P}}\tilde{\Omega}.$$

Then by

$$2(\beta\lambda - \epsilon) - m = \tilde{F_0} \cdot \tilde{\Omega} \geq \tilde{m}$$

we see that

$$m + \tilde{m} \leq 2(\beta\lambda - \epsilon).$$

Using $\tilde{m} \leq m$, we then get

$$\tilde{m} \leq \lambda\beta - \epsilon.$$

By our construction, the log pair

$$(\tilde{S}, (1 - \beta + \epsilon)\tilde{C} + a\tilde{F_0} + \tilde{\Omega} + (a + m - \beta + \epsilon)G)$$

is not lc at some point $Q \in G$. Using inversion of adjunction along the exceptional curve G, it is easy to find that

$$Q = \tilde{P}.$$

Now let $h:\ \hat{S} \to \tilde{S}$ be the blow up of P_1 and let H be the exceptional curve of h. We let $\hat{C}$, $\hat{F_0}$, $\tilde{G}$ and $\hat{\Omega}$ be the proper transform of C, $\tilde{F_0}$, G and $\tilde{\Omega}$ respectively on the surface $\tilde{S}$. Let us set

$$\hat{P} = \hat{C} \cap H.$$

By our construction, the log pair

$$(\hat{S}, (1 - \beta + \epsilon)\hat{C} + a\hat{F_0} + \hat{\Omega} + (a + m - \beta + \epsilon)\tilde{G} + (2a + m + \tilde{m} - 2\beta + 2\epsilon)H)$$

is not lc at some point $O \in H$. Using inversion of adjunction along the exceptional curve H, it is easy to find that

$$O = \hat{P}.$$

So we see that, the log pair

$$(\hat{S}, (1 - \beta + \epsilon)\hat{C} + \hat{\Omega} + (2a + m + \tilde{m} - 2\beta + 2\epsilon)H)$$

is not lc at the point $\hat{P}$. We put

$$\hat{m} = \text{mult}_{\hat{P}}\hat{\Omega}.$$

It is clear that

$$\hat{m} \leq (H \cdot \hat{\Omega})_{\hat{P}} \leq \tilde{m}_2 \leq (\lambda\beta - \epsilon).$$

Meanwhile, by $m + \tilde{m} \leq 2(\lambda\beta - \epsilon)$ and $2a \leq 1 - \frac{\beta}{5}$, it is easy to check that

$$(2a + m + \tilde{m} - 2\beta + 2\epsilon) \leq 1 - \frac{\beta}{10}.$$

Then we can apply Theorem 2.16 at the point $\hat{P}$. We get

$$(\hat{C} \cdot \hat{\Omega})_{\hat{P}} > \frac{(H \cdot \hat{\Omega})_{\hat{P}}}{(H \cdot \hat{\Omega})_{\hat{P}} - (\beta - \epsilon)}(1 - (2a + m + \tilde{m} - 2\beta + 2\epsilon)) - (\beta - \epsilon).$$

So we have

$$(\hat{C} \cdot \hat{\Omega})_{\hat{P}} > \frac{(\lambda\beta - \epsilon)}{(\lambda\beta - \epsilon) - (\beta - \epsilon)} \cdot \frac{\beta}{10} - (\beta - \epsilon) = \frac{(\lambda\beta - \epsilon)}{10(\lambda - 1)} - (\beta - \epsilon).$$

Now simply using

$$(\hat{C} \cdot \hat{\Omega})_{\hat{P}} \leq C \cdot \Omega - m - \tilde{m} \leq C \cdot \Omega = 2\lambda - (\lambda\beta - \epsilon)(r - 4) \leq 2\lambda,$$

we get

$$2\lambda > \frac{(\lambda\beta - \epsilon)}{10(\lambda - 1)} - (\beta - \epsilon).$$

Using $0 \leq \epsilon \leq \frac{2\beta}{3}$, we arrive at

$$2\lambda + \beta > \frac{(\lambda - \frac{2}{3})}{10(\lambda - 1)}\beta,$$

which gives a contradiction since we chose $\lambda = 1 + \frac{\beta}{100}$ with β sufficiently small. The proof of Theorem 5.6 is complete.

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